Splitting the Birthday Cake

This week's [Fiddler on the Proof](#) (1 March 2024) asks:

> You and two friends all have March birthdays, so you've decided to celebrate together with one big cake that has delicious frosting around its perimeter. To share the cake fairly, you want to ensure that (1) each of you gets the same amount of cake, by area, and (2) each of you gets the same amount of frosting along the cake's edge.
>
> What's more, you want to cut the cake by starting at a single point inside of it, and then making three straight cuts to the edge from that point. You've already worked out ways to do this for circular and square cakes, as shown below.
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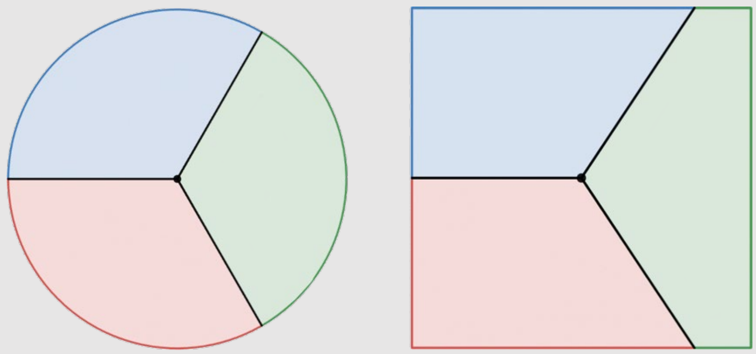
> 
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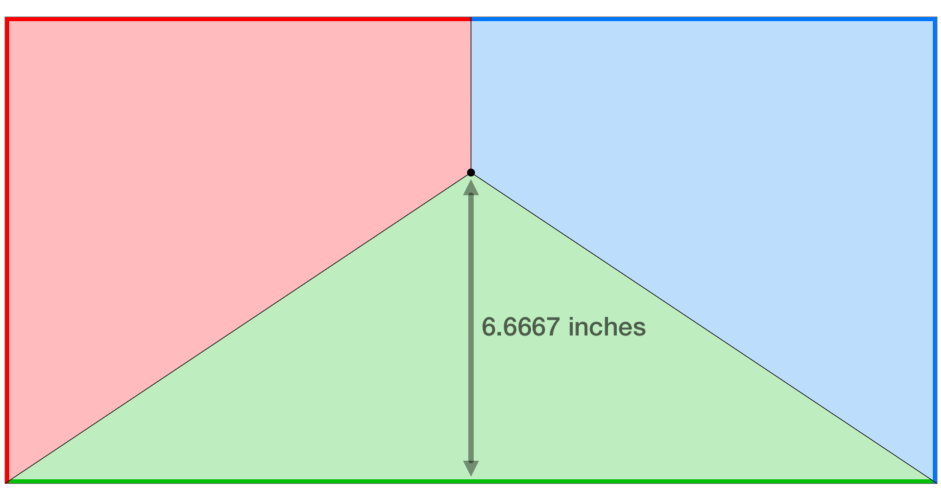
> However, the cake you bought is rectangular, with a length of 20 inches and a width of 10 inches. Using the coordinate system of your choice, describe a way this particular cake can be cut fairly, so that all three of you get the same amount in terms of both area and the cake's perimeter. Again, there should be three straight cuts emanating from a single point inside the cake.

Each friend's piece should include 20 inches of frosting and 66.667 square inches of cake. An easy way to achieve this is to have one of the pieces be a triangle with its base along one of the long sides of the cake. Then we calculate the required height for this triangle. The formula for area of a triangle

$$\frac{base \cdot height}{2} = area$$

implies the height must be 6.6667 inches. If we make the triangle symmetric, then it is possible to divide the rest of the cake into two equal pieces.



Extra Credit

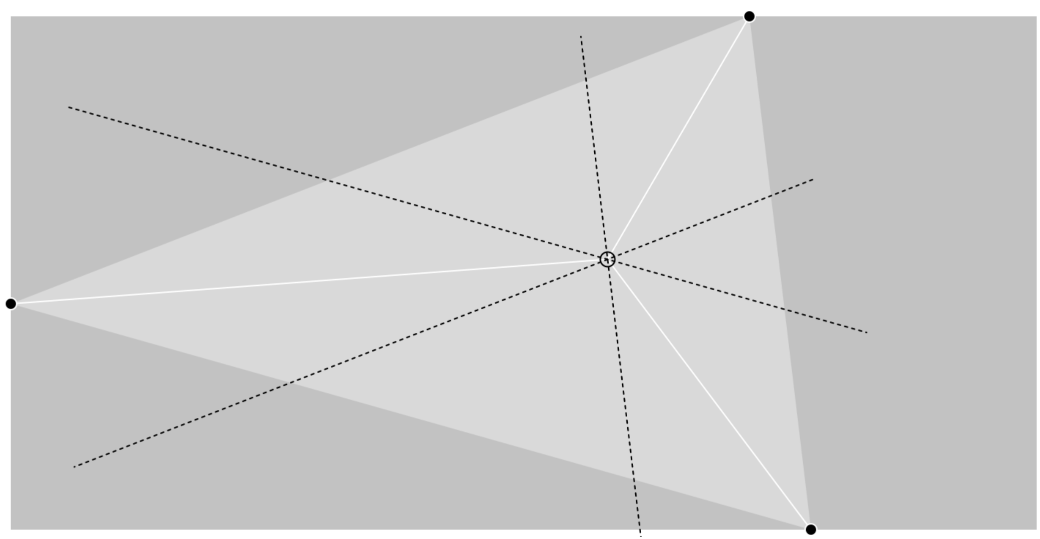
By now, you've hopefully found at least one central point from which you can make your three straight cuts, so that you and your two friends each get the same amount of birthday cake, in terms of both area and the frosting along the edge.

As it turns out, there are many possible central points. Together, these points form a locus that's a closed shape. For Extra Credit, what is the area contained within this shape?

The problem now is a more general one. For a given set of exterior points where the cutlines will meet the edge of the cake, we want to find the corresponding central point representing the other end of the straight cuts. The three exterior points are related to each other in that they are all 20 inches apart from each other along the perimeter of the cake.

Let's look at the exterior points in pairs. Every pair of exterior points share a 20-inch perimeter of frosting between them. And that length of frosting is attached to a piece of cake defined by those points. We want to find out what possible central points give this piece is correct area (one third of the whole cake).

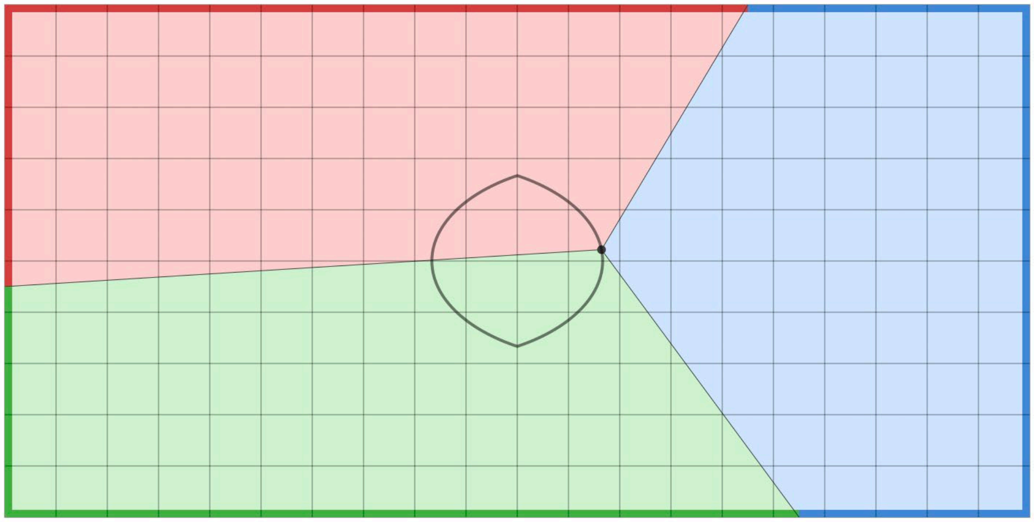
For each pair of points, there is a portion of the cake that must be included in the piece defined by those points. (Those portions are shown in dark gray in the diagram below.) In addition, each piece of cake includes a triangle that is defined by the two exterior points and the common central point.



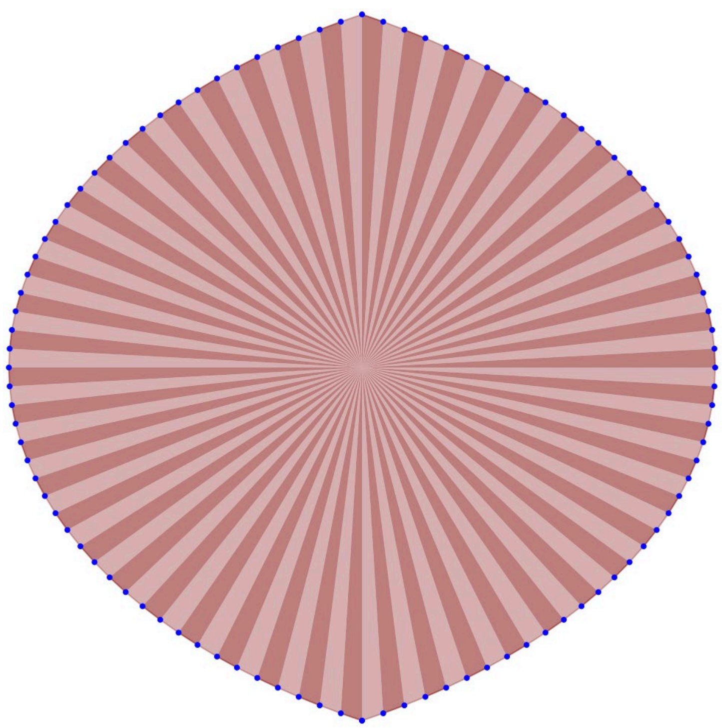
We know that the area of this triangle plus the area of the dark gray portion must equal 66.667 square inches. If the base of a triangle is the line joining two exterior points, we can figure out the required height of the triangle. Knowing the triangle's height gives us a line that the common central point must lie on.

The dotted lines show possible positions of the apexes of each triangle for the triangles to have the correct height. Amazingly, the dotted lines all meet each other at the same point. That tells us where the common central point must be.

I wrote a computer program to calculate the central point associated with a given triad of outside points. Here is a plot of the positions where the various central points lie.



The points form a goofy bulb shape. I estimated the area of the bulb by dividing it into narrow triangles that radiate out from the center of the cake.



This diagram shows the bulb divided into 100 narrow triangles. The total area of the triangles is 7.8759821670756 square inches.

This is slightly less than the correct area of the bulb because using triangles assumes straight lines between the dots on the edge of the bulb whereas the bulb actually curves out slightly.

Using more triangles gives a more accurate answer:

Number of triangles	Calculated area
10	7.4836213340368
100	7.8759821670756
1,000	7.8799551543847
10,000	7.8799948892609
100,000	7.8799952866102
Exact area*	**7.8799952906238**

* The exact area above is provided for comparison and is taken from the formula derived here by Emilie Mitchell:

$$\text{area} = \frac{50(4\pi\sqrt{3} - 9)}{81}$$