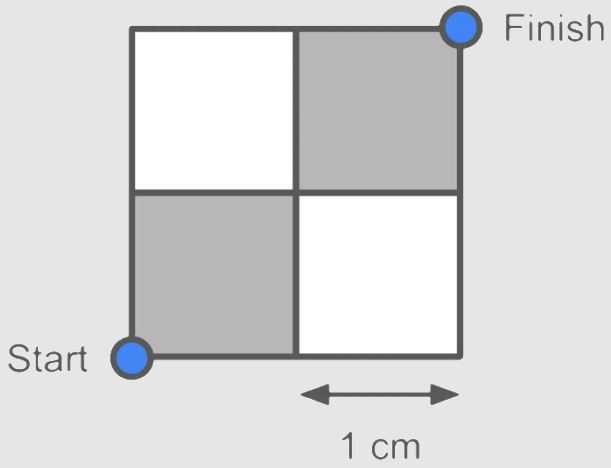
Chessboard Race

This week's Fiddler on the Proof (26 January 2024) asks:

A very tiny Alice (not to be confused with the Alice from last week) is racing across a 2-by-2 chessboard, as shown below, where each smaller square has a side length of 1 cm. Alice starts at the bottom-left corner of the bottom-left black square and is trying to reach the top-right corner of the top-right black square.

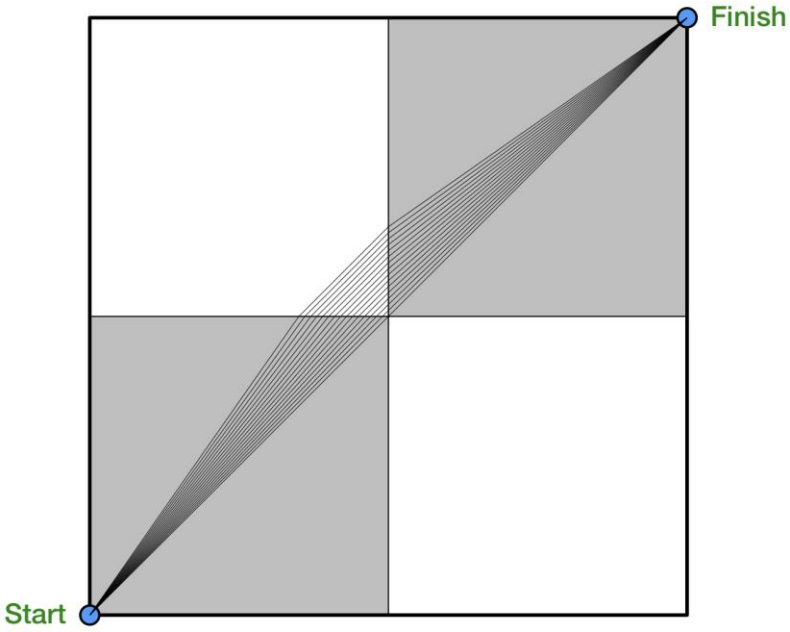


There's just one catch. Alice moves faster on the white squares than she does on the black squares. Her speed on the white squares is 1 cm per minute (again, she's *very* small), while her speed on the black squares is 0.9 cm per minute.

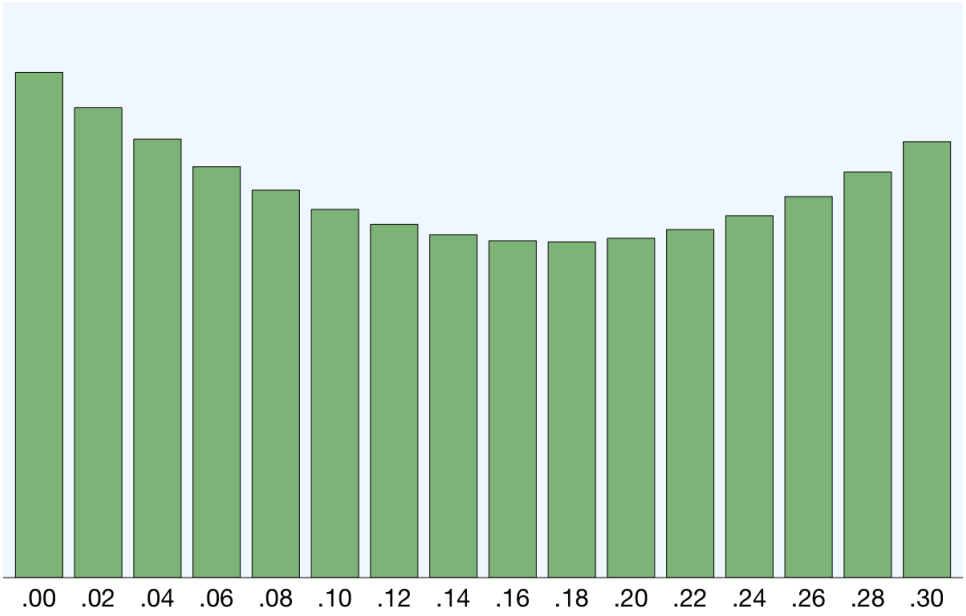
What's the least amount of time it will take her to reach the finish?

Deviating from a Straight Line

Alice has competing desires. She would like to walk in a straight line to minimize her distance. But she also wants to spend as little time as possible on the black squares. Here is the range of her best choices:



The following plot shows the time it takes Alice to walk each of the paths illustrated above.



The shortest time occurs at a deviation of 0.18 cm. That is to say, Alice should aim for a point 0.18 centimeters to the left of the center of the board where she will switch from black to white. (Likewise 0.18 centimeters above the center of the board for her switch from white to black.)

If she does this, her time spent walking on the first black square is

$$\frac{\sqrt{0.82^2 + 1^2}}{0.9} = 1.43690$$

Her time spent walking on the middle white square is

$$\sqrt{0.18^2 + 0.18^2} = 0.25456$$

And her time spent walking on the last black square is also 1.43690.

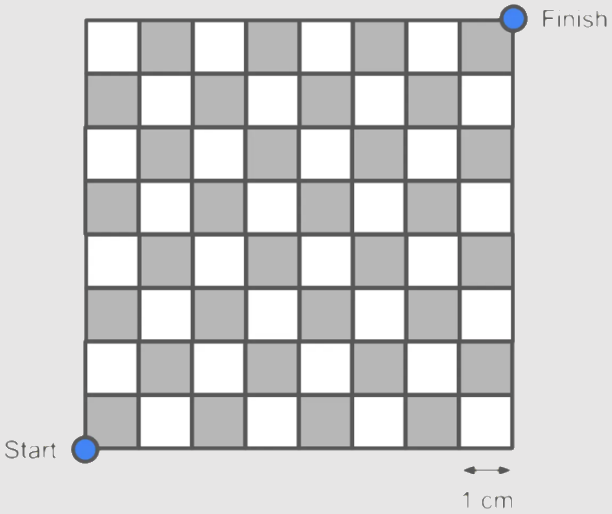
Adding these together gives

$$1.43690 + 0.25456 + 1.43690 = 3.12836$$

Alice can get from start to finish in about 3.12836 minutes.

Extra Credit

Instead of a 2-by-2 chessboard, Alice now wants to traverse an 8-by-8 chessboard, as shown below. Again, she's going from the bottom-left corner of the bottom-left black square to the top-right corner of the top-right black square.



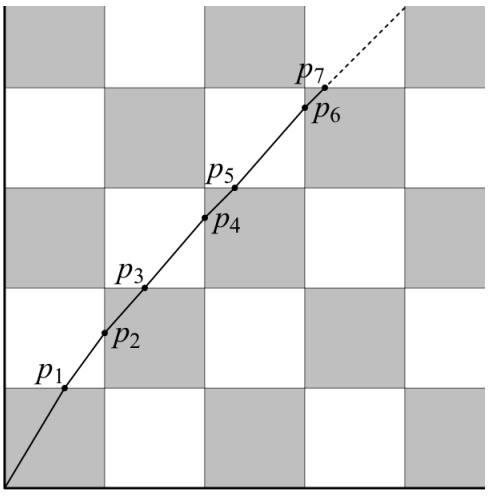
As before, her speed on the white squares is 1 cm per minute, while her speed on the black squares is 0.9 cm per minute.

What's the least amount of time it will take her to reach the finish?

The chessboard is $\sqrt{128} \cong 11.3137$ centimeters along the diagonal, so 11.3137 minutes is the least time Alice might use. (Since Alice must start and finish in black squares, this lower limit obviously can't be achieved.)

If Alice stayed on the diagonal, and spent all her time walking on black squares, the time she would need is $\frac{\sqrt{128}}{0.9} \cong 12.5708$ minutes. This is an upper limit on the time it will take here to traverse the board. She can probably do better by deviating from the diagonal to spend more time walking on those glorious white squares.

I imagined Alice walking on alternating colored squares, gradually spending more time on white squares than black, along a path that looks something like this:



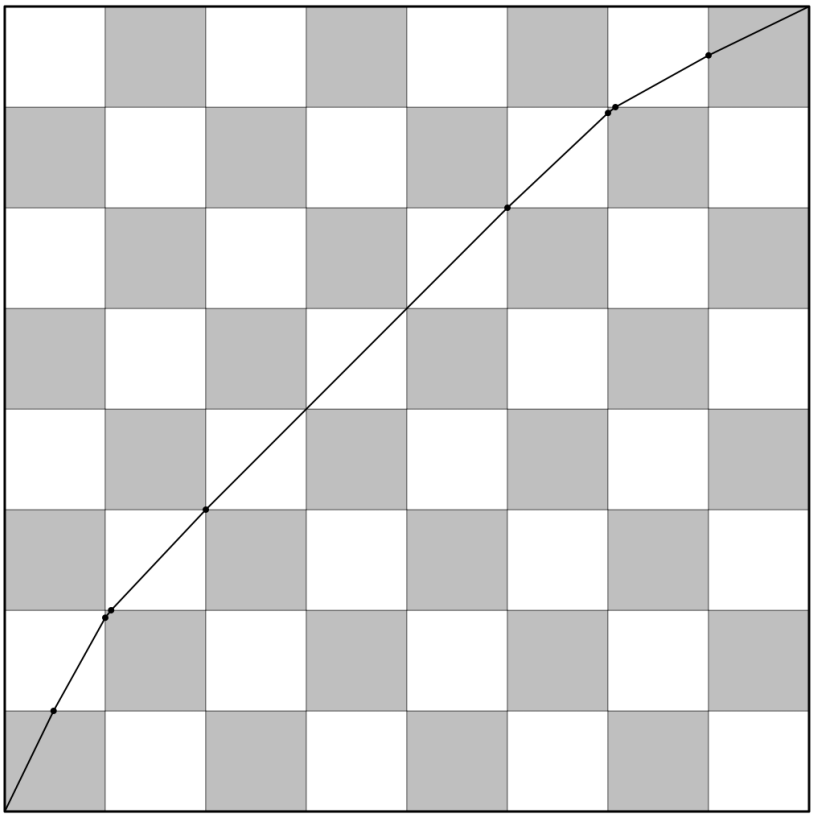
(The last half of the path is a mirror image of the first half.)

I labeled the points each where Alice changes the color of square she is walking on. Since Alice walks in a straight line within a square, these seven points define Alice's path.

So what are the best positions for these points? I approached this question by placing the p's at arbitrary positions, with the odd-numbered *p*'s having a fixed integer *y*-component and the even-numbered *p*'s having a fixed integer *x*-component.

Then I *fiddled* with the positions of the points. If moving a point in one direction made Alice's time longer, I tried moving it in the other direction. By doing this over and over, adjusting a different point each time, I progressively made Alice's time better.

Here is what Alice's final path looks like:



It's amazing how quickly it is worth it to Alice to be completely on the white squares. Using this route, Alice walks a total of 11.5202487 centimeters but only 21 percent of her route is on black squares.

How Tall is Alice?

Let's estimate that if Alice were 2 meters tall she would be able to walk 6 km/h. We are told Alice actually walks 1 cm/min. That's 600,000 cm in one hour versus 60 cm in one hour, or a ratio of 10,000 to 1.

Alice's actual height must be somewhere in around 2 meters/10,000, or about 200 microns.