In Search of Equitable Dice

This week's Fiddler on the Proof (26 July 2024) asks:

Suppose you (player A) and a friend (player B) are playing a game in which you alternate rolling a die. So the order of play is AB|AB|AB, and so on. (The vertical bars here are just for organizational purposes, and do not signify anything special that happens.) The first player to roll a five wins the game. As it turns out, whoever goes first has a distinct advantage!

What about other ways you and your friend could take turns, ways that might result in a fairer game? For example, consider the "snake" method, in which the order is reversed after each time you both roll: AB|BA|AB|BA, and so on.

Assuming you are the first to roll, what is the probability you will win the game?

Extra Credit

Another way to take turns is to use the Thue-Morse sequence, where the *entire* history of the order is reversed after each round. As an illustration, consider the first few rounds:

- Round 1: Player A goes first.
- Round 2: Only A went in the first round. So now player B goes.
- Round 3: Up to this point, the order has been AB. Reversing this, round 3's order is BA.
- Round 4: Up to this point, the order has been ABBA. Reversing this, round 4's order is BAAB.
- Round 5: Up to this point, the order has been ABBABAAB. Reversing this, round 5's order is BAABABBA.

Writing this out as a single sequence of turns, the order is A|B|BA|BAAB|BAABABBA, and so on.

Assuming you are the first to roll, what is the probability you will win the game?

Each method described above is fairer than the previous method, but the player who goes first has a slight advantage in all three cases. That got me wondering, is there some sequence of turns that is totally fair to both sides?

After playing around some, I found the answer is "yes, there are many completely fair sequences." However, as far as I could tell: (1) There are no *finite* sequences of turns that work. And (2) there is no "easy to describe" sequence. Which is to say, all the sequences I found exhibit some randomness.

Here are two examples.

Narrow Path

Somebody has to go first, so I arbitrarily decided that A goes first. Then I repeatedly applied the following rule:

> After each turn, the next player to roll is the one who has the lower probability of having won the game up to this point.

For example, after A rolls, A's probability of having won is 1/6 and B's probability of having won is 0 (because he hasn't rolled yet). So B goes next.

Now B's probability of having won is $\frac{5}{6} \cdot \frac{1}{6} = \frac{5}{36}$. This is still less than than A's probability of having won, so B rolls again.

After B's second roll, his probability of having won is $\frac{55}{216}$, which is now bigger that A's probability, so it is A's turn again. And it goes on like this. Here's what the pattern looks like for the first 500 rolls.

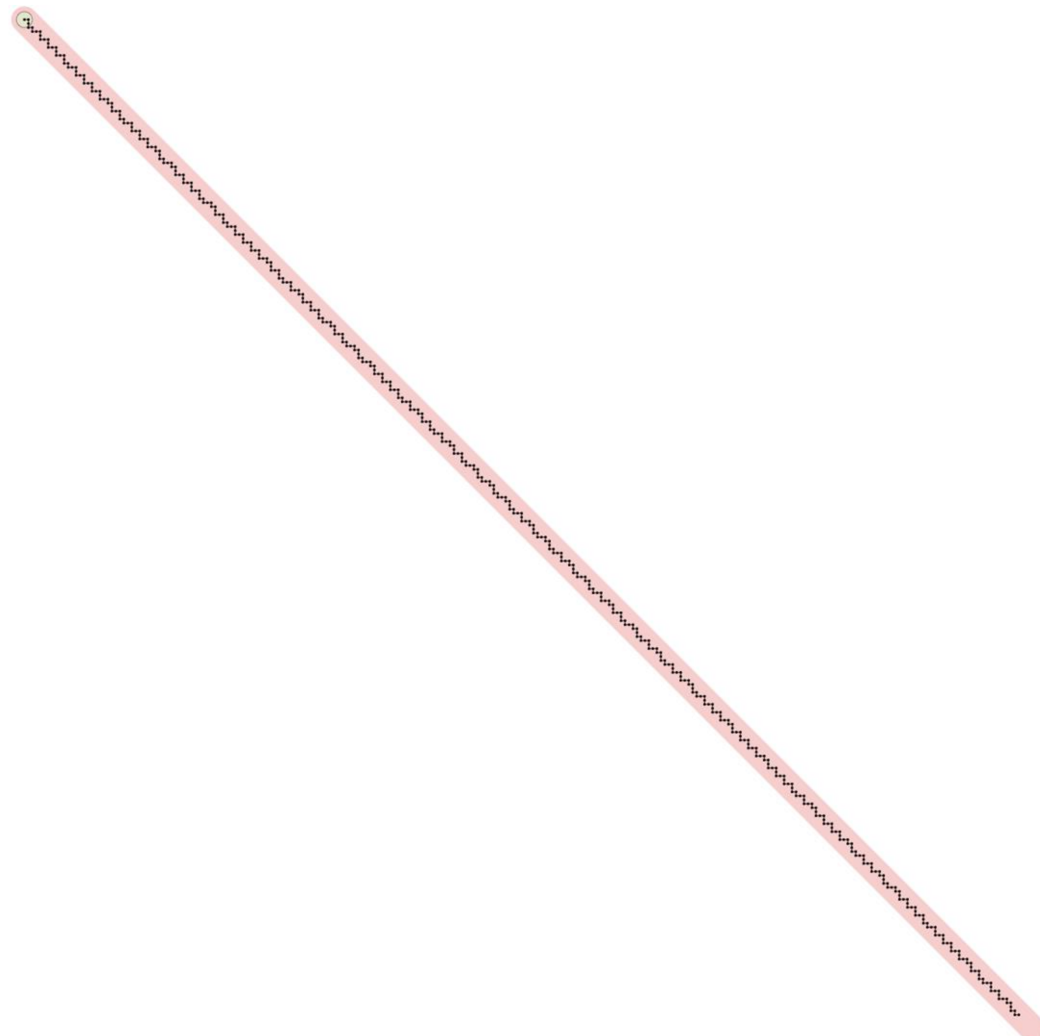
**A BB A B AA BB AA BB AA BB AA BB A B AA BB AA BB AA BB AA BB AA B A BB AA BB A B AA BB AA BB AA
BB AA B A BB A B AA B A BB AA BB AA BB AA BB A B AA B A BB AA BB A B AA BB AA BB AA BB AA BB AA B
A BB AA BB A B AA BB AA B A BB AA BB AA BB A B AA B A BB AA BB A B AA B A BB A B AA B A BB AA BB
AA BB AA BB A B AA BB AA B A BB AA BB A B AA BB AA B A BB AA BB A B AA B A BB A B AA B A BB A B AA
B A BB A B AA BB AA B A BB A B AA B A BB A B AA BB AA B A BB A B AA B A BB A B AA BB AA B A BB AA
BB A B AA B A BB A B AA BB AA B A BB A B AA BB AA BB AA B A BB A B AA BB AA BB AA BB AA B A BB AA
BB AA BB AA BB AA B A BB AA BB AA BB AA BB A B AA BB AA B A BB AA BB AA BB AA BB AA B A BB A B
AA BB AA BB AA B A BB A B AA B A BB AA BB AA BB AA BB A B AA BB AA BB AA BB AA BB AA B A BB AA
BB A B AA BB AA BB AA B A BB A B A**

There are 311 turns, for an average turn length of 1.608 rolls. [After 1 million rolls, the average turn length is 1.6239. I wonder what constant is being converged on here?]

After 500 rolls:

A's probability of having won is	0.4999999999999999999999999999999998670478
B's probability of having won is	0.4999999999999999999999999999999998762811
Probability of no winner yet is	0.0000000000000000000000000000000002566711

Here is a graphical representation of the sequence of turns. The graph starts in the upper left. Each roll by A is indicated by a tiny move to the right. Each roll by B is indicated by a tiny move down.



The graph follows a fairly narrow path, close to a straight line that goes down at a 45° angle.

Wide Path

This time I used the following rule:

> After each roll, the same player rolls again as many times as he can without putting his chances of having won over 50%. Otherwise the turn changes.

A goes first, after which his probability of having won is 1/6, which is less than ½. So A rolls again, making his probability of having won $\frac{1}{6} + \frac{5}{6} \cdot \frac{1}{6} = \frac{11}{36}$, still less than ½. After A's third roll, his probability of having won is $\frac{91}{216}$, still less than ½.

If A were to roll a fourth time, his probability of winning would be $\frac{671}{1296}$, which is more than ½. We can't let that happen or else B would never be able to catch up. So, after three rolls by A, it is B's turn.

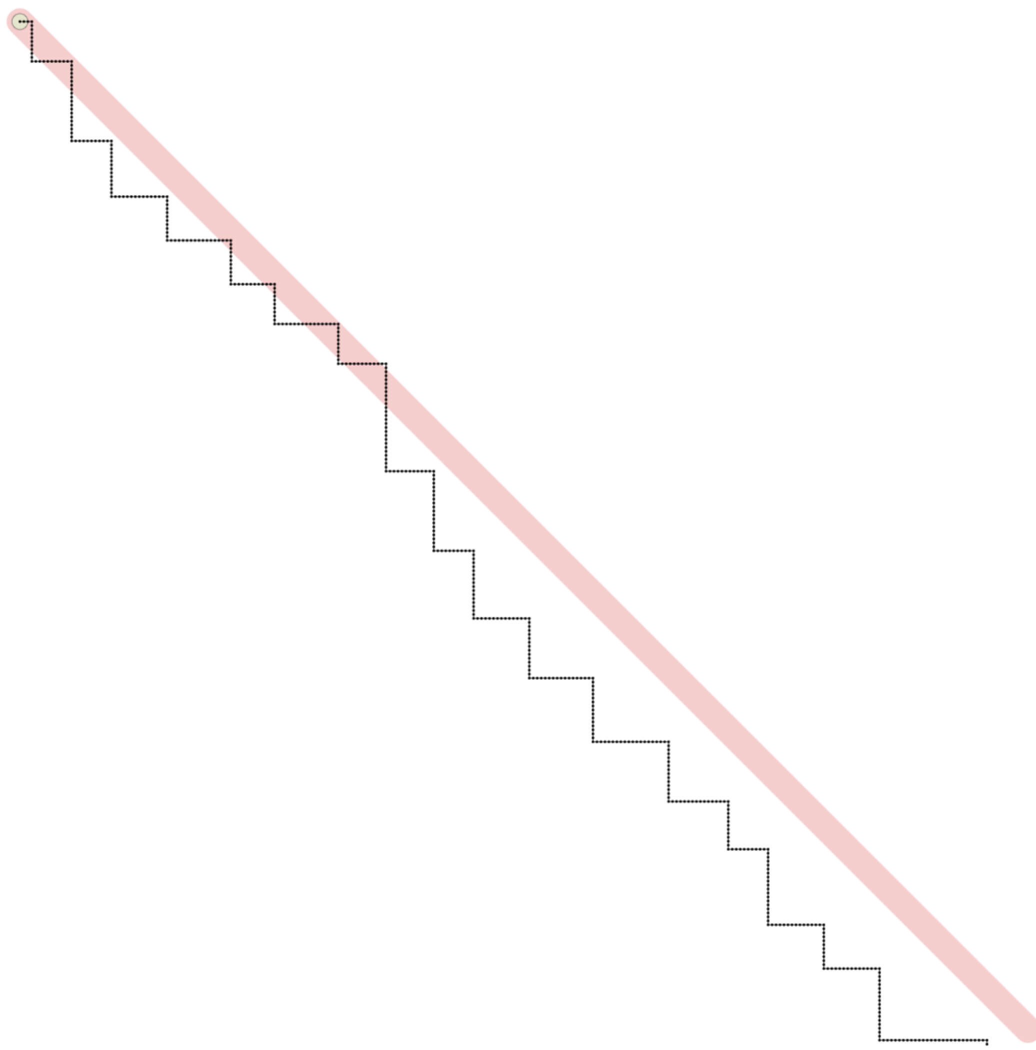
Now B rolls as many times in a row as he can without putting his own chances of having won over 50%. It turns out that B can roll 10 times. And so it goes, back and forth, with neither side ever quite reaching 50% winning chances. Here is what the first 500 rolls look like:

AAA BBBBBBBBBB AAAAAAAAAA BBBBBBBBBBBBBBBBBBBB AAAAAAAAAA BBBBBBBBBBBBBB AAAAAAAAAAAAAA BBBBBBBBBBB AAAAAAAAAAAAAAAA BBBBBBBBBBB AAAAAAAAAAA BBBBBBBBBB AAAAAAAAAAAAAAAA BBBBBBBBBB AAAAAAAAAAAA BBBBBBBBBBBBBBBBBBBBBBBBBBB AAAAAAAAAAAA BBBBBBBBBBBBBBBBBBBB AAAAAAAAAA BBBBBBBBBBBBBBBBB AAAAAAAAAAAAAA BBBBBBBBBBBBBBB AAAAAAAAAAAAAAAA BBBBBBBBBBBBBBBB AAAAAAAAAAAAAAAAAAA BBBBBBBBBBBBBBB AAAAAAAAAAAAAAA BBBBBBBBBBBB AAAAAAAAAA BBBBBBBBBBBBBBBBBBB AAAAAAAAAAAAAA BBBBBBBBBBB AAAAAAAAAAAAAA BBBBBBBBBBBBBBBBBB AAAAAAAAAAAAAAAAAAAAAAAAAAA B

There are 36 turns, for an average turn length of 13.9 rolls. [The average turn length after 1 million rolls is 14.664.] After 500 rolls, we have:

A's probability of having won is	0.49999999999999999999999999999999999999738247
B's probability of having won is	0.49999999999999999999999999999999999997695042
Probability of no winner yet	0.00000000000000000000000000000000000002566711

Here is the graphical representation of the sequence of turns.



This time the graph follows quite a zig-zaggy path. It deviates from the 45° line and wavers quite a bit.

Fairest Game After N Rolls

Suppose the players decide to stop the game after *n* rolls even if there is no winner. What sequence of turns gives the fairest result? By "fairest," I mean that the chances of winning for the two players are as close to each other as possible. Here are the results I found for values of *n* from 1 to 40.

		Probability of Winning		
n	**Pattern**	**Player A**	**Player B**	**Difference**
1	A	.16666666667	.00000000000	**.1666666666666667**
2	AB	.16666666667	.13888888889	**.0277777777777778**
3	ABB	.16666666667	.25462962963	.0879629629629630
4	ABBA	.26311728395	.25462962963	**.0084876543209876**
5	AABBB	.30555555556	.29256687243	.0129886831275720
6	ABBABA	.33009687929	.33500514403	**.0049082647462277**
7	AABBBBA	.36137188500	.35954646776	**.0018254172382259**
8	ABABBBAA	.38473734473	.38269461591	.0020427288142051
9	AABABBBBB	.40200617284	.40418712769	.0021809548531728
10	AABBBABABB	.41904875876	.41944565835	**.0003968995823384**
11	AABBABBABBB	.43244467783	.43296733643	.0005226585962154
12	ABABBBAABAAB	.43244467783	.44388728676	**.0000687716899959**
13	ABABBABAABBBA	.45335472631	.45318139471	.0001733316006964
14	ABBAABBAAABBBB	.46106886270	.46104457148	**.0000242912180836**
15	ABABBAABBBAABBA	.46753335460	.46756117388	.0000278192826847
16	ABAABBBBBABABAAA	.47296645828	.47294564879	.0000208094944634
17	ABBABABABBAAABAAA	.47746551277	.47746124312	**.0000042696582415**
18	AABBABBBBAABBAABBA	.48122039851	.48121856474	**.0000018337689096**
19	ABAABBBBAAABBBBBBAB	.48435028566	.48434885037	**.0000014352913453**
20	ABABABBBABBAABAAABAA	.48695867362	.48695727307	**.0000014005545095**
21	AABBABABBBBAABBBBBABB	.48913167895	.48913160996	**.0000000689932346**
22	AABBBABABBAABBABBBBAAB	.49094293405	.49094314004	.0000002059915102
23	AABBBAABBBBABABAABABBBB	.49245252466	.49245253708	**.0000000124221051**
24	AABBBAABBBABBAABBBABBABB	.49371044293	.49371044186	**.0000000010710147**
25	ABBBAAABABABAABABABBBAABB	.49475869063	.49475871336	.0000000227356233
26	ABABBBBAAAAABBABBAAAAABAAA	.49563225394	.49563224938	**.0000000045637820**
27	ABABBBAABABAAABBABBABAABBAB	.49636021287	.49636020656	.0000000063097916
28	ABBBAAABAABBBAABABABAABABABB	.49696684118	.49696684168	**.0000000004964424**
29	AABBBABABABABBBBBABABBAABAAAA	.49747236674	.49747236898	.0000000022490556
30	ABBABABAAAABBBABBBBBBAABBBAAAB	.49789364031	.49789363946	.0000000008538803
31	ABAABBBBABABBBABAAABABAAAAAABAA	.49824470008	.49824469973	**.0000000003484549**
32	AABBBBBAAAABBBABABABAAAAAAABBABA	.49853724979	.49853725005	**.0000000002632986**
33	ABABAABBBBBABABBABAABABBABAABAAAB	.49878104159	.49878104161	**.0000000000282263**
34	ABBABABAABABBABAABBAAABBABAAAABAAA	.49898420134	.49898420133	**.0000000000094079**
35	ABBABBAAAAABBBABBABABBBAAAABABABABB	.49915350115	.49915350107	.0000000000852033
36	AABBBBBABAAAABBAAAAABBABABBBABABAAAB	.49929458425	.49929458426	.0000000000103723
37	ABBAABABBBAABABBABAABAAABBAAABBABAAAA	.49941215355	.49941215355	**.0000000000005418**
38	ABBABBAAABBABAAABAAAABBABAAAAABBAABABA	.49951012796	.49951012796	.0000000000021470
39	ABAABBBAABBBABBBBBBAABAABBBBBBBAABBBAAA	.49959177330	.49959177330	.0000000000010829
40	ABABBABBAAAABBBABAABBBBABBBBBABBAAAAABAB	.49965981108	.49965981108	**.0000000000004546**

It's remarkable just how random these patterns are! The fact that they all begin with A was my choice. Each pattern has a counterpart beginning with B that is equally good. If you ignore the leading A's, the remaining letters don't seem to have any pattern to them at all.

Extra Credit Solution

Back to the question posed in the Extra Credit.

Define an ordered pair (a, b) representing respectively: A's probability of having won and B's probability of having won. Now define a "squared" function that performs the following operation:

$$\text{squared}(a, b) \;=\; (a, b) + (b, a) \cdot (1 - \; a - \; b)$$

It calculates the probabilities associated with inverting and appending to the sequence so far. It flips the order of the probabilities , multiplies both probabilities by the probability of not having a winner yet, and adds the result to the probabilities we have computed so far.

Start with the simplest pair of probabilities we know of – those resulting from A rolling the die once. A has 1/6 chance of winning; B has no chance of winning. So the initial order pair is ordered pair (1/6, 0).

Now apply the "squared" function to the ordered pair over and over again. After just 14 self applications of the this function, the probabilities are accurate to 1,000 decimal places. The a component of the final ordered pair gives the answer to the puzzle:

0.50159033920426902511418551480717180281201968982185029937945292096699864915203068623747928199190515709213319100609037669546306241835077663115724847708794145699430441887046490391374183519279414810428347882214957973242538180697801494291186939537533556217859941153999712650939096902960400255467615993140625137951580361234980493166842479431086469782633379721747806694971224276956440971974920340005336856477322055690577269868605628620354961655523852198343520120278682598536675592843690762839058139174023934080163668141523574440526940241671672754820736021655410799909416065951924853830205236645936832870744886864831998157262543530777069106714708368812103975976749334779172192598079387141346832159254564201118669509493448261234910867839523053191689639433695072893169897256171784453202433341651470323263904250034622449354708667034700853138030084664131767305043292445335208518425583731790525365564600844162732601650690994941224395162359861825886744327808536672436623037661363981942127722234438683496911466164 68