Factorial Number System

This week's Fiddler on the Proof (6 October 2023) asks:

> Number systems like decimal, binary, and hexadecimal each have a fixed number of allowed digits in each place (10, 2, and 16, respectively), while "mixed-radix" systems like the 24-hour clock can have a different number of allowed digits in each place (24 hours, 60 minutes, and 60 seconds, for example). Counting works the same way in all of these systems: When incrementing the rightmost digit causes it to roll from its maximum value back to zero, we "carry" this overflow by incrementing the next digit to the left: 09 increments to 10 in base 10, and 00:00:59 increments to 00:01:00 on the clock.
>
> In the factorial number system (not a prank), the Nth digit from the right has base N, and thus a maximum allowable digit of $N-1$. As a result, the rightmost digit is always 0 (don't get me started on "base 1"), the digit to its left can be 0 or 1 (in good old base 2), the next digit can be 0, 1, or 2 (in base 3), and so on. This is called the factorial number system because the place value of the N^{th} digit is $(N-1)!$.
>
> Let's look at an example. Suppose a number is written **103210** in the factorial number system. What is this number in base 10? It's **1**(5!) + **0**(4!) + **3**(3!) + **2**(2!) + **1**(1!) + **0**(0!), or 143. (Note that the numbers in **red** are exactly the digits from the factorial number representation.)
>
> What is the smallest number in the factorial number system that is divisible by the whole numbers from 1 through 5, while also containing all of the digits 0 through 5? (Note that this number can have more than one copy of any given digit.)

Terminology and Aesthetics

I will use the term **factoriad** to mean "a number expressed in the factorial number system."

Somehow I find it more pleasing to omit the superfluous final digit 0 in factoriads, so that is what I will do for the rest of this article. (It makes no difference to the puzzle.)

Divisibility

There are two contraints we have to worry about: (1) Divisibility and (2) inclusivity. Let's look at the divisibility requirement first.

The lowest common multiple of the numbers 1 through 5 is 60. The first few factoriads that are mulitples of 60 look like this:

	10!	9!	8!	7!	6!	5!	4!	3!	2!	1!	← Place value
Decimal	**10**	**9**	**8**	**7**	**6**	**5**	**4**	**3**	**2**	**1**	← Max digit
60							2	2	0	0	
120						1	0	0	0	0	
180						1	2	2	0	0	
240						2	0	0	0	0	

You can see that the least significant 4 digits alternate between “0,0,0,0” and “2,2,0,0”. In fact, any factoriad that has one of these two endings is a multiple of 60 and therefore satisfies the divisibility requirement.

Inclusivity

We also want our factoriad to include the digits 1, 2, 3, 4, and 5 in its representation. Of the two endings available, “2,2,0,0” is better because it already includes one of the needed digits, namely 2.

Now we just have to fill in the remaining digits. We’ll use 1 in the leftmost column because it has the smallest value, then 3 in the next column because it has the next smallest value, etc.

Answer

The smallest factoriad that fits the criteria is

	10!	9!	8!	7!	6!	5!	4!	3!	2!	1!	← Place value
Decimal	**10**	**9**	**8**	**7**	**6**	**5**	**4**	**3**	**2**	**1**	← Max digit
58,980			1	3	4	5	2	2	0	0	

Extra Credit

What is the smallest number in the factorial number system that contains each of the digits 1 through 9 *exactly once*, and is also divisible by all of the whole numbers from 1 through 9? (Note that this number can contain any amount of zeros.)

The lowest common multiple of the numbers 1 through 9 is 2,520. The first few factoriads that are mulitples of 2,520 look like this:

	10!	9!	8!	7!	6!	5!	4!	3!	2!	1!	← Place value
Decimal	**10**	**9**	**8**	**7**	**6**	**5**	**4**	**3**	**2**	**1**	← Max digit
2,520					3	3	0	0	0	0	
5,040				1	0	0	0	0	0	0	
7,560				1	3	3	0	0	0	0	
10,080				2	0	0	0	0	0	0	

The endings alternate between “3,3,0,0,0,0” and “0,0,0,0,0,0”. We can’t use “3,3,0,0,0,0” as an ending because this version of the puzzle requires all the digits of the factoriad to be different. That leaves “0,0,0,0,0,0” as the only ending possible.

So now all we have to do is fill in the most signficant places with the digits 1 through 9. Here goes:

15!	14!	13!	12!	11!	10!	9!	8!	7!	6!	5!	4!	3!	2!	1!	← Place value
15	**14**	**13**	**12**	**11**	**10**	**9**	**8**	**7**	**6**	**5**	**4**	**3**	**2**	**1**	← Max digit
1	2	3	4	5	6	7	8	9	0	0	0	0	0	0	

But now we have a problem; this factoriad is malformed! There is a 9 digit in the 7!s column and the maximum digit allowed for that column is 7. To turn this into a legitimate factoriad, we'll have to rearrange some of the digits.

15!	14!	13!	12!	11!	10!	9!	8!	7!	6!	5!	4!	3!	2!	1!	← Place value
15	**14**	**13**	**12**	**11**	**10**	**9**	**8**	**7**	**6**	**5**	**4**	**3**	**2**	**1**	← Max digit
1	2	3	4	5	6	9	8	7	0	0	0	0	0	0	

And that is the answer.

Extra Credit Extended

The following table shows the smallest factoriad that is divisible by all the digits 1 through *n* while using each of those digits exactly once in its representation.

Digits	24!	23!	22!	21!	20!	19!	18!	17!	16!	15!	14!	13!	12!	11!	10!	9!	8!	7!	6!	5!	4!	3!	2!	1!
	24	**23**	**22**	**21**	**20**	**19**	**18**	**17**	**16**	**15**	**14**	**13**	**12**	**11**	**10**	**9**	**8**	**7**	**6**	**5**	**4**	**3**	**2**	**1**
1																								1
1 to 2																						1	2	0
1 to 3																				1	2	3	0	0
1 to 4																			1	3	4	2	0	0
1 to 5																1	2	3	4	5	0	0	0	0
1 to 6															1	2	3	4	6	5	0	0	0	0
1 to 7												1	2	3	4	5	6	7	0	0	0	0	0	0
1 to 8											1	2	3	4	5	6	8	7	0	0	0	0	0	0
1 to 9										1	2	3	4	5	6	9	8	7	0	0	0	0	0	0
1 to 10									1	2	3	4	5	6	10	9	8	7	0	0	0	0	0	0
1 to 11								1	2	3	4	6	8	11	10	9	7	5	0	0	0	0	0	0
1 to 12							1	2	3	4	6	8	12	11	10	9	7	5	0	0	0	0	0	0
1 to 13						1	2	3	4	6	11	13	12	10	9	8	7	5	0	0	0	0	0	0
1 to 14					1	2	3	4	6	11	14	13	12	10	9	8	7	5	0	0	0	0	0	0
1 to 15				1	2	3	4	6	11	15	14	13	12	10	9	8	7	5	0	0	0	0	0	0
1 to 16			1	2	3	4	6	11	16	15	14	13	12	10	9	8	7	5	0	0	0	0	0	0
1 to 17		1	2	3	4	10	12	17	16	15	14	13	9	11	8	5	7	6	0	0	0	0	0	0
1 to 18	1	2	3	4	10	12	18	17	16	15	14	13	9	11	8	5	7	6	0	0	0	0	0	0

(The digits that use their column's maximum value are highlighted in green.)