Racing to Get Pancakes

This week's Fiddler on the Proof (19 January 2024) asks:

> Three siblings are at a playground: Alice, Bob, and Carey. Alice, the oldest, gets a call from their dad—their pancake dinner is ready! But they won't get to eat until all three kids are home.
>
> They each walk home at a different constant speed. Alice can walk home in 10 minutes, Bob can do it in 20, and Carey in 30. Fortunately, any of the kids can carry any of the others on their back without reducing their own walking speed. (However, none of them can carry a kid who is, in turn, carrying another kid.) Assume that they can pick someone up, set someone down, and change direction instantaneously.
>
> What is the fastest they can all get home? (Your answer should be in minutes.)

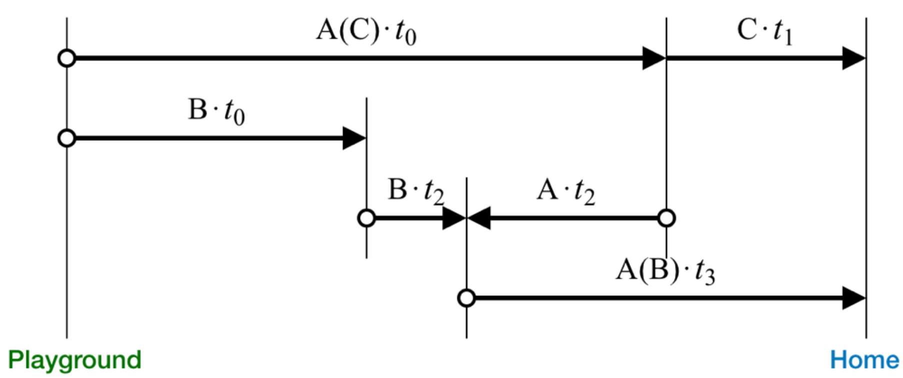
How Fast Does Each Child Walk?

The puzzle doesn't say how far the playground is from home. To make the calculations easier, let's imagine the playground as being 1 mile from home. The actual distance doesn't matter because the "miles" will cancel out in the end.

Alice walks home in 10 minutes, so let A = 6 mph = Alice's walking speed.
Bob walks home in 20 minutes, so let B = 3 mph = Bob's walking speed.
Carey walks home in 30 minutes, so let C = 2 mph = Carey's walking speed.

First Idea

Suppose Alice carries Carey most of the way home, then goes back to meet Bob and carries Bob home while Carey finishes the trip on her own. Here is what that looks like:



The capital letters represent who is walking. Parentheses mean that child is being carried.

The t's represent the duration of each walking segment. They are the unknowns we are trying to figure out.

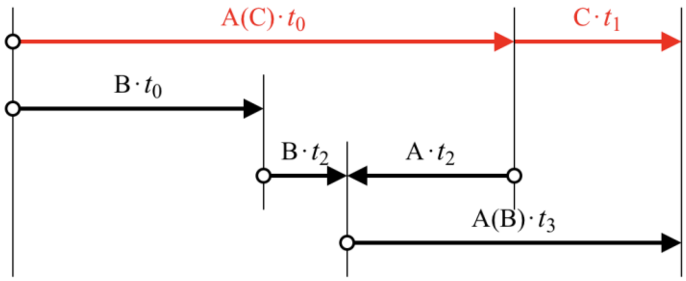
- t_0 is how long Alice carries Carey,
- t_1 is how long it takes Carey to finish getting home on her own,
- t_2 is how long it takes Alice to go back and meet Bob, and

- t_3 is how long it takes Alice to carry Bob the rest of the way home.

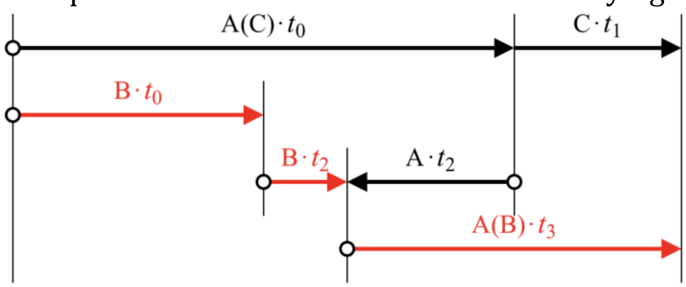
The distance represented by each arrow is the speed of the walker times the duration of that segment.

Here are some observations we can make:

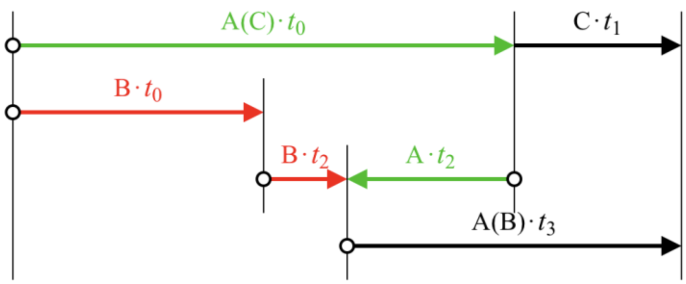
1. The distance Alice carries Carey plus the distance Carey walks on her own is 1 mile.



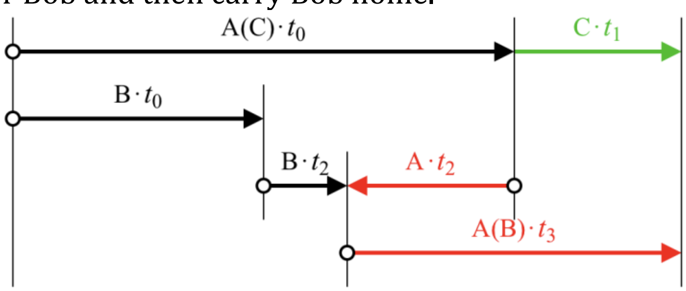
2. The distance Bob walks while Alice is carrying Carey plus the distance Bob walks while Alice comes back for him plus the distance Alice walks while carrying Bob is 1 mile.



3. The distance Alice carries Carey less the distance Alice walks back to meet Bob equals the distance Bob walks while Alice carries Carey plus the distance Bob walks after that to meet Alice.



4. The time it takes Carey to walk the rest of the way home is the same as the time it takes Alice to walk back for Bob and then carry Bob home.



We can express these observations as equations:

$$6 \cdot t_0 + 2 \cdot t_1 = 1$$
$$3 \cdot t_0 + 3 \cdot t_2 + 6 \cdot t_3 = 1$$
$$6 \cdot t_0 - 6 \cdot t_2 = 3 \cdot t_0 + 3 \cdot t_2$$
$$t_1 = t_2 + t_3$$

That's four equations and four unknowns. Solving for the t's gives:

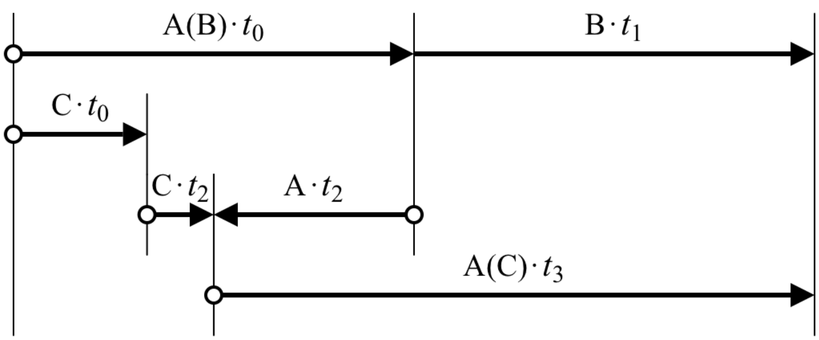
$$t_0 = \frac{1}{8} \quad t_1 = \frac{1}{8} \quad t_2 = \frac{1}{24} \quad t_3 = \frac{1}{12}$$

The time it takes Carey to get home is $t_0 + t_1 = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$ hour.
The time it takes Bob to get home is $t_0 + t_2 + t_3 = \frac{1}{8} + \frac{1}{24} + \frac{1}{12} = \frac{1}{4}$ hour.
The time it takes Alice to get home is $t_0 + t_2 + t_3 = \frac{1}{8} + \frac{1}{24} + \frac{1}{12} = \frac{1}{4}$ hour.

All three children can be home in **15 minutes**.

Another Idea

What if Alice carries Bob part of the way home, then goes back to get Carey and carries her the rest of the way home while Bob finishes on his own?



System of equations:

$$6 \cdot t_0 + 3 \cdot t_1 = 1$$
$$2 \cdot t_0 + 2 \cdot t_2 + 6 \cdot t_3 = 1$$
$$6 \cdot t_0 - 6 \cdot t_2 = 2 \cdot t_0 + 2 \cdot t_2$$
$$t_1 = t_2 + t_3$$

Gives:

$$t_0 = \frac{1}{12} \quad t_1 = \frac{1}{6} \quad t_2 = \frac{1}{24} \quad t_3 = \frac{1}{8}$$

The time it takes Carey to get home is $t_0 + t_2 + t_3 = \frac{1}{12} + \frac{1}{24} + \frac{1}{8} = \frac{1}{4}$ hour.
The time it takes Bob to get home is $t_0 + t_1 = \frac{1}{12} + \frac{1}{6} = \frac{1}{4}$ hour.
The time it takes Alice to get home is $t_0 + t_2 + t_3 = \frac{1}{12} + \frac{1}{24} + \frac{1}{8} = \frac{1}{4}$ hour.

Again, everyone is home is **15 minutes**. Both ideas work equally well.

Extra Credit

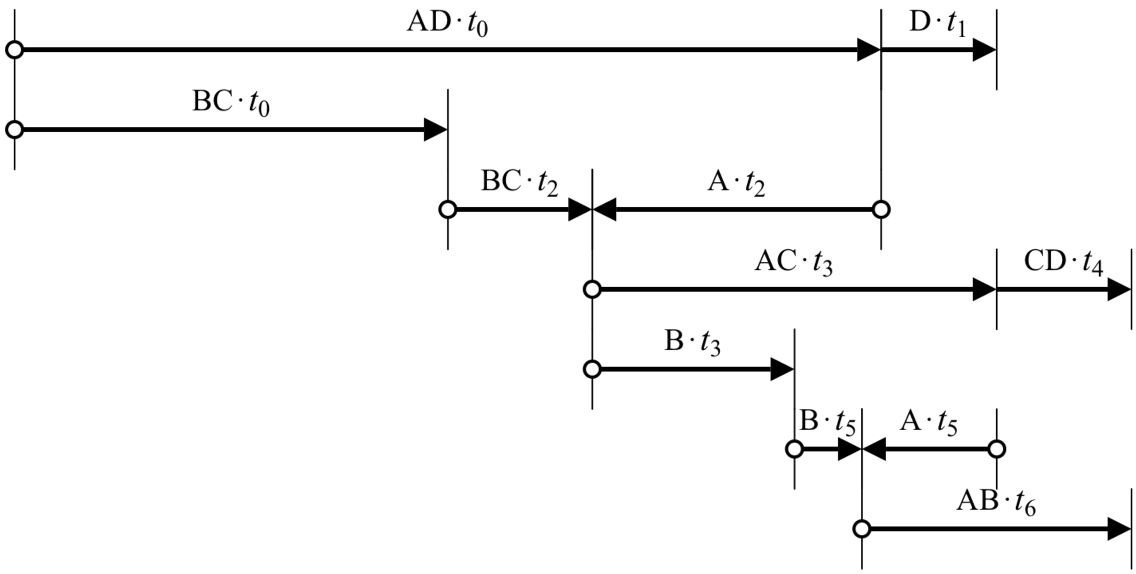
Suppose Alice, Bob, and Carey have a fourth sibling named Dee, who is with them at the playground. Dee is the slowest, needing 60 minutes to walk home. As before, any kid can carry any other kid, and they won't start eating until everyone is home.

What is the fastest they can all get home? (Your answer should be in minutes.)

This is quite a bit tougher because of all the possibilities of who carries who. First, let's work out the value of D, the speed that Dee walks:

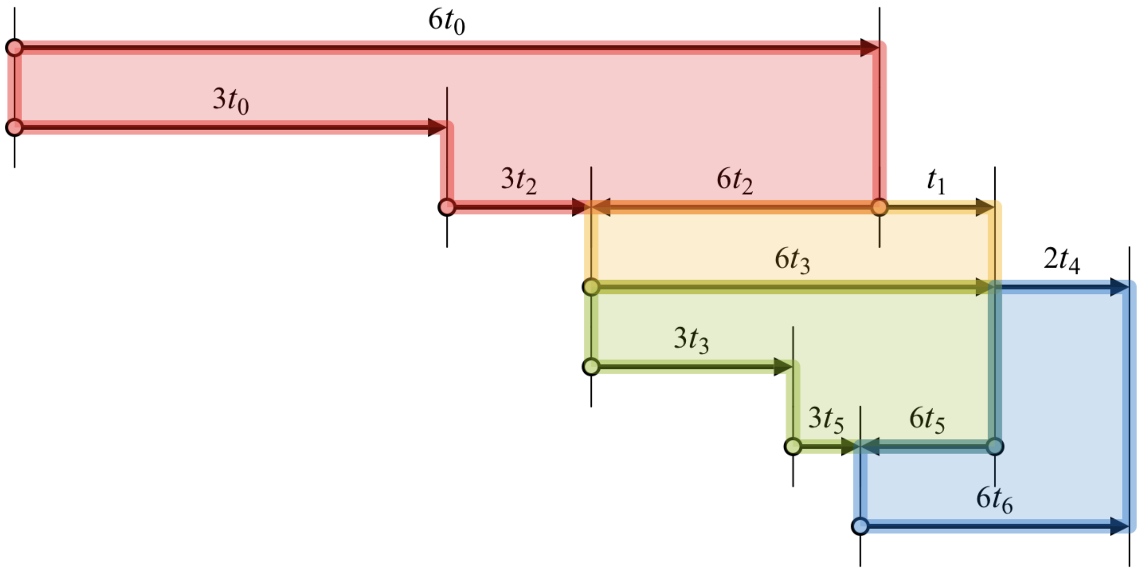
Dee walks home in 60 minutes, so D = 1 mph.

After some playing around, here's one idea I came up with. Not sure if it's the best.



- Alice carries Dee for t_0 hours, then Dee continues on his own for t_1 hours.
- While that's happening Bob carries Carey for the same t_0 hours that Alice was carrying Dee. Bob continues forward while Alice backtracks for t_2 hours whereupon they meet.
- Alice takes Carey from Bob and continues toward home for t_3 hours. There they run into Dee and Carey continues toward home, now carrying Dee, for t_4 hours.
- Meanwhile Alice backtracks for t_5 hours where she meets Bob. Alice carries Bob the rest of the way home, which takes t_6 hours.

If we replace the letters in the above diagram with the speeds they represent, the diagram looks like this:



Examining the chart for loops, we can deduce some apparent equivalancies:

$$6t_0 = 3t_0 + 3t_2 + 6t_2$$
$$6t_2 + t_1 = 6t_3$$
$$6t_3 = 3t_3 + 3t_5 + 6t_5$$
$$6t_5 + 2t_4 = 6t_6$$

We know that the entire journey is one mile. Tracking the path home for, say, Dee gives:

$$6t_0 + t_1 + 2t_4 = 1$$

And we want all four children to arrive home at the same time. Equalizing the total time used by Dee, and Carey, and Bob gives this compound equation:

$$t_0 + t_1 + t_4 = t_0 + t_2 + t_3 + t_4 = t_0 + t_2 + t_3 + t_5 + t_6$$

(The time intervals for Alice are the same as for Bob.)

That's seven equations and seven unknowns. Solving them gives:

$$t_0 = \frac{15}{116} \quad t_1 = \frac{3}{29} \quad t2 = \frac{5}{116} \quad t_3 = \frac{7}{116} \quad t_4 = \frac{7}{116} \quad t_5 = \frac{7}{348} \quad t_6 = \frac{7}{174}$$

We can now follow the path that each child takes to get home and see how long they take.

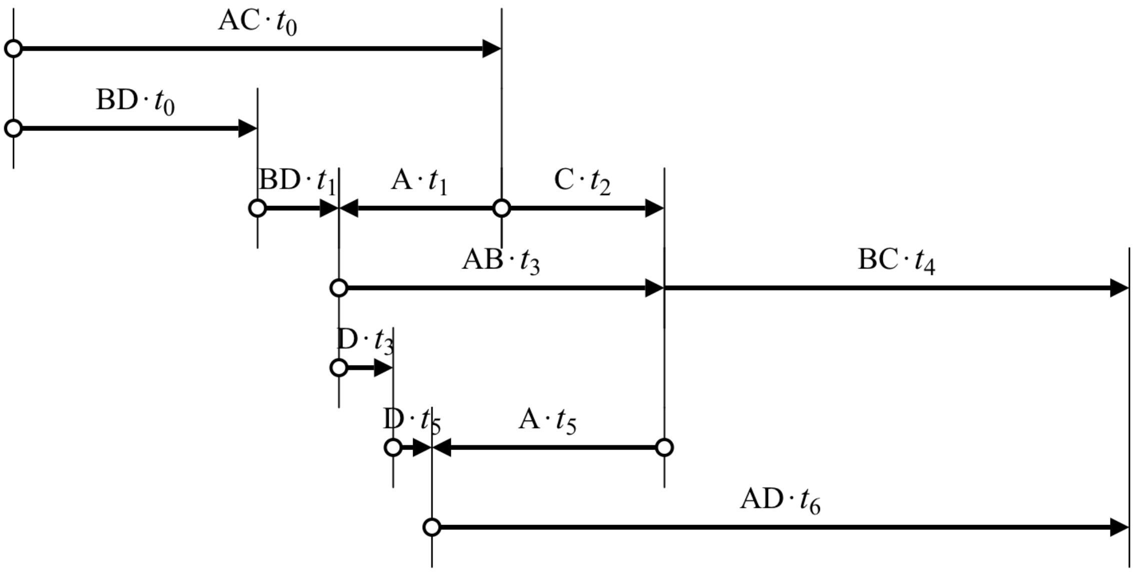
It takes Dee $t_0 + t_1 + t_4 = \frac{15}{116} + \frac{3}{29} + \frac{7}{116} = \frac{17}{58}$ hours.
It takes Carey $t_0 + t_2 + t_3 + t_4 = \frac{15}{116} + \frac{5}{116} + \frac{7}{116} + \frac{7}{116} = \frac{17}{58}$ hours.
It takes Bob $t_0 + t_2 + t_3 + t_5 + t_6 = \frac{15}{116} + \frac{5}{116} + \frac{7}{116} + \frac{7}{348} + \frac{7}{174} = \frac{17}{58}$ hours.
It takes Alice $t_0 + t_2 + t_3 + t_5 + t_6 = \frac{15}{116} + \frac{5}{116} + \frac{7}{116} + \frac{7}{348} + \frac{7}{174} = \frac{17}{58}$ hours.

All four children arrive home in $\frac{17}{58}$ hours, or about **17.5862 minutes**.

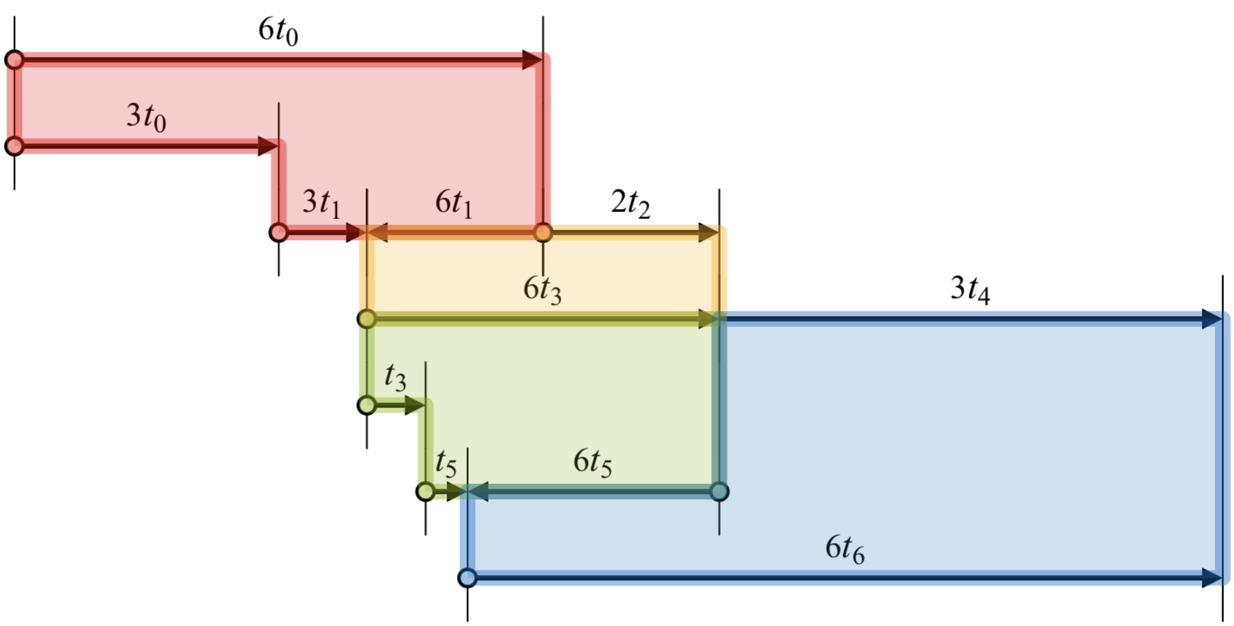
That was a lot of work. Time for pancakes!

Postscript

On [Laurent Lessard's]() website, **Sanandan Swaminathan** offers an even more inventive solution:



- Alice carries Carey, and Bob carries Dee for a while.
- Then Alice leaves Carey and walks back to Bob and Dee; meanwhile, Carey continues on her own.
- When Alice meets Bob, Alice carries Bob toward home, leaving Dee on his own.
- When Alice and Bob meet Carey, Bob carries Carey the rest of the way home.
- While Bob's doing that, Alice goes back to get Dee.
- When Alice meets Dee, Alice carries Dee the rest of the way home.



Equations deduced from the cycles above:

$$6t_0 = 3t_0 + 3t_1 + 6t_1$$

$$6t_1 + 2t_2 = 6t_3$$
$$6t_3 = t_3 + t_5 + 6t_5$$
$$6t_5 + 3t_4 = 6t_6$$

Distance Carey travels to get home:

$$6t_0 + 2t2 + 3t4 = 1$$

All children arrive home at the same time (Dee's time = Carey's time = Bob's time):

$$t_0 + t_1 + t_3 + t_5 + t_6 = t_0 + t_2 + t_4 = t_0 + t_1 + t_3 + t_4$$

Seven equations, seven unknowns give:

$$t_0 = \frac{7}{96} \quad t_1 = \frac{7}{288} \quad t_2 = \frac{7}{96} \quad t_3 = \frac{7}{144} \quad t_4 = \frac{5}{36} \quad t_5 = \frac{5}{144} \quad t_6 = \frac{5}{48}$$

Dee gets home in $\frac{7}{96} + \frac{7}{288} + \frac{7}{144} + \frac{5}{144} + \frac{5}{48} = \frac{41}{144}$ hours.
Carey gets home in $\frac{7}{96} + \frac{7}{96} + \frac{5}{36} = \frac{41}{144}$ hours.
Bob gets home in $\frac{7}{96} + \frac{7}{288} + \frac{7}{144} + \frac{5}{36} = \frac{41}{144}$ hours.
Alice gets home in $\frac{7}{96} + \frac{7}{288} + \frac{7}{144} + \frac{5}{144} + \frac{5}{48} = \frac{41}{144}$ hours.

All children arrive home in $\frac{41}{144}$ hours, or about **17.0833 minutes**.