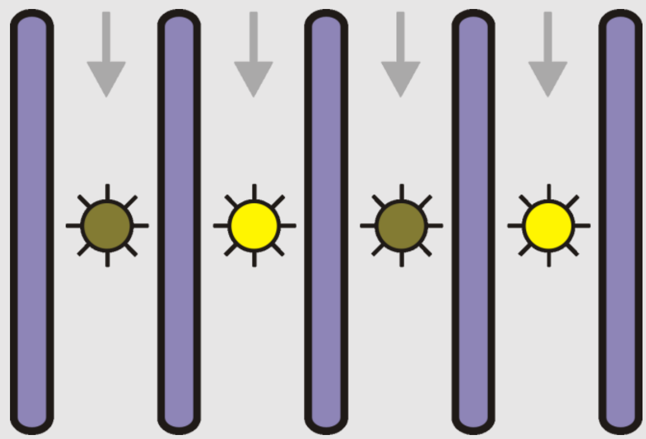
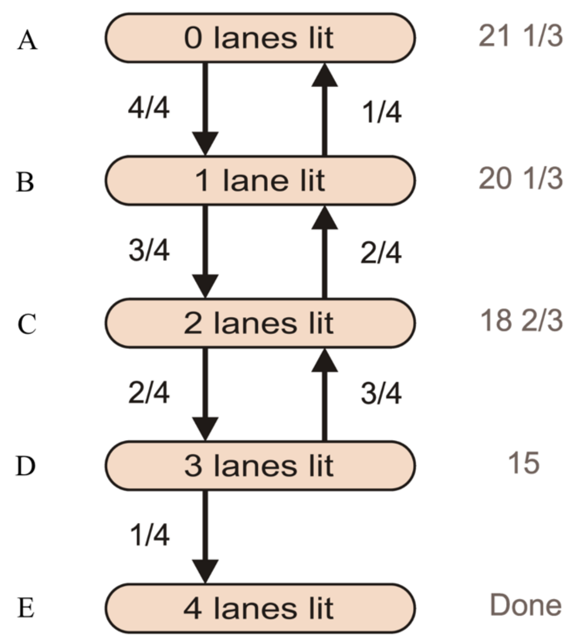
Light Up the Pinball Lanes

This week's Fiddler on the Proof (29 Sept 2023) asks:

> You're playing a game of pinball that includes four lanes, each of which is **initially unlit**.
>
> Every time you flip the pinball, it passes through exactly one of the four lanes (chosen at random) and toggles that lane's state. So if that lane is unlit, it becomes lit after the ball passes through. But if the lane is lit, it becomes unlit after the ball passes through.
>
> 
>
> On average, how many times will you have to flip the pinball until all four lanes are lit?

There are 5 states that we need to worry about: no lanes lit, one lane lit, two lanes lit, three lanes lit, and four lanes lit. The following diagram shows the relationship between these states.



For example, if there is 1 lane lit you have a 25% chance of returning to "0 lanes lit" when a ball passes through and a 75% chance of going to "2 lanes lit."

Using a Spreadsheet

Perhaps the easiest way to calculate the expected number of flips is to use a spreadsheet.

You'll need five cells, one for each number of lanes lit. Enter the formulas for these relationships into the spreadsheet (remembering to add 1 each time you make a transition).

Each cell will calculate average number of flips required to turn all the lights on from the state it represents. Those averages are shown down the right side of the diagram above.

System of Equations

Alternatively, you can set up a system of equations representing the relationship between the states. Let A be the expected flips required from "no lanes lit," B be the expected flips required from "1 lane lit," etc. Then we have the following equations:

$$A = B + 1$$

$$B = \frac{A}{4} + \frac{3C}{4} + 1$$

$$C = \frac{B}{2} + \frac{D}{2} + 1$$

$$D = \frac{3C}{4} + \frac{E}{4} + 1$$

$$E = 0$$

The last equation ($E = 0$) represents the state where all 4 lanes are lit. Since we are done at that point, the expected number of flips needed is zero.

Five equations and five unknowns lead to the following solutions:

$$A = \frac{64}{3}, B = \frac{61}{3}, C = \frac{56}{3}, D = 15, E = 0$$

It takes $\frac{64}{3}$, or $21\frac{1}{3}$ flips on average to turn the lights in all 4 lanes on.

Extra Credit

Instead of four lanes, now suppose your pinball game has *N* lanes. And let's say that *E*(*N*) represents the average number of pinball flips it takes until all *N* lanes are lit up.
Now, each time you increase the number of lanes by one, you find that it takes you approximately twice as long to light up all the lanes. In other words, *E*(N+1) seems to be about double *E*(*N*).

But upon closer examination, you find that it's not *quite* double. Moreover, there's a particular value of *N* where the ratio *E*(*N*+1)/*E*(*N*) is at a minimum. What is this value of *N*?

N	N+1	E(N+1)	E(N)	E(N+1)/E(N)
0	1 lane	1.00000 flip		
1	2 lanes	4.00000 flips	1.00000 flip	4.000000
2	3 lanes	10.0000 flips	4.00000 flips	2.500000
3	4 lanes	21.3333 flips	10.0000 flips	2.133333
4	5 lanes	42.6667 flips	21.3333 flips	2.000000
5	6 lanes	83.2000 flips	42.6667 flips	1.950000
6	7 lanes	161.067 flips	83.2000 flips	1.935897
7	8 lanes	312.076 flips	161.067 flips	1.937559
8	9 lanes	607.085 flips	312.076 flips	1.945313
9	10 lanes	1186.54 flips	607.085 flips	1.954485
10	11 lanes	2329.19 flips	1186.54 flips	1.963014
11	12 lanes	4588.94 flips	2329.19 flips	1.970183