Simplified Backjack

This week's [Fiddler on the Proof](#) (24 May 2024) Extra Credit asks:

> You're playing a modified version of blackjack, where the deck consists of exactly 10 cards numbered 1 through 10. Unlike traditional blackjack, in which the ace can count as 1 or 11, the 1 here always has a value of 1.
>
> You shuffle the deck so the order of the cards is completely random, after which you draw one card at a time. You keep drawing until the sum of your drawn cards is at least 21. If the sum is *exactly* 21, you win! But if the sum is greater than 21, you "bust," or lose.
>
> You decide to be risk averse. Whenever there's even the *slightest* chance you could bust on the next card, you quit the round and start over.
>
> On average, how many rounds should you expect to start until you finally win?

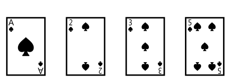
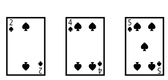
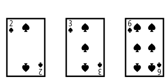
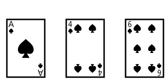
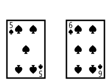
Penultimate Positions

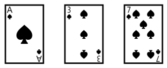
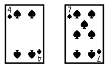
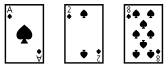
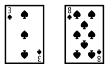
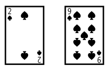
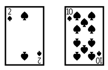
In order to win using the risk-averse strategy, you first need to reach a "penultimate" position. This is a position from which you can win on the next card but is also risk-free in that there is no card remaining in the deck that would put you over 21.

I found 11 penultimate positions, which are listed below. For each position, it is easy to calculate the chance of reaching it. Suppose n cards have been played. There are $\binom{10}{n}$ combinations that have that number of cards, so the chance that these particular cards are the ones that appeared is $1/\binom{10}{n}$.

Having reached a penultimate position, there is only one card that will win on the next draw. The chance that of getting that card is $1/(10-n)$.

Together, the chance of reaching a particular penultimate position and then winning from there is the product of these two probabilities.

Cards drawn	Number of cards n	Chance of reaching $1/\binom{10}{n}$	Chance of winning from $1/(10-n)$	Combined probability
	4	$1/210$	$1/6$	$1/1260$
	3	$1/120$	$1/7$	$1/840$
	3	$1/120$	$1/7$	$1/840$
	3	$1/120$	$1/7$	$1/840$
	2	$1/45$	$1/8$	$1/360$

	3	1/120	1/7	1/840
	2	1/45	1/8	1/360
	3	1/120	1/7	1/840
	2	1/45	1/8	1/360
	2	1/45	1/8	1/360
	2	1/45	1/8	1/360
			Total	**13/630**

The chance of winning on any given round is 13/630.

Number of Rounds

Let E be the expected number of rounds we will need to win. There is a 13/630 chance of winning on the first try and 617/630 that we will have to try again. This implies

$$E = \frac{13}{630} + \frac{617}{630} \cdot (1 + E)$$

Solving for E gives

$$E = \frac{630}{13} \cong 48.46$$

It will take, on average, **48.46 rounds** to win the game using the risk-averse strategy.