Smallest Largest Triangle

This week's Fiddler on the Proof (5 January 2024) asks:

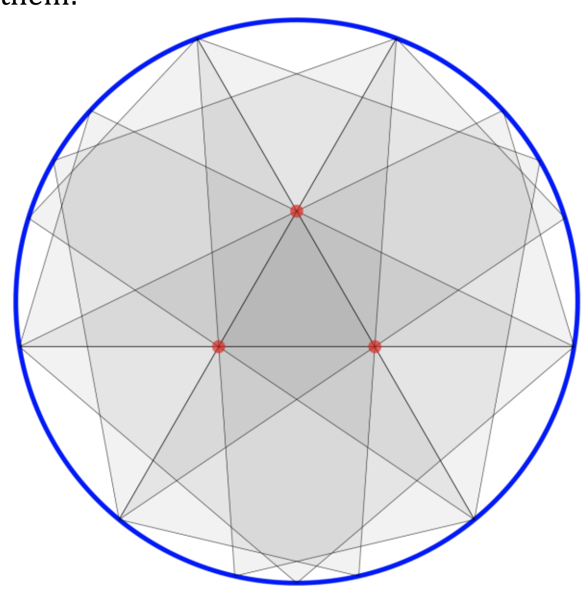
> There are countless ways to pick three points within the unit disk (i.e., the points on the unit circle and inside the unit circle). And for each of these ways, there are countless triangles you can draw within the disk that do not strictly contain any of the three points. By "strictly," I mean that the three points can be on the edges of a triangle, but not *inside* a triangle.
>
> For any trio of points in the unit disk, consider the *largest* triangle (by area) within the disk that doesn't strictly contain any of the three points.
>
> By positioning the three points strategically, it's possible to reduce the area of the largest such triangle. What is the *minimum* possible area this largest triangle can have?

There are two things we have to do here. First, for a given set of three points, we have to figure out the largest triangle that avoids those points. That's no easy feat since there are so many ways to place triangles.

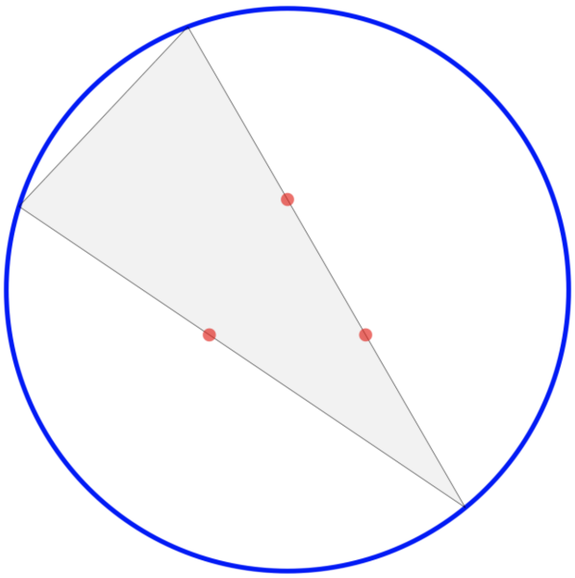
But even if we had a magic function that found the largest fitting triangle, how do we find the correct position for the dots? That's not easy either.

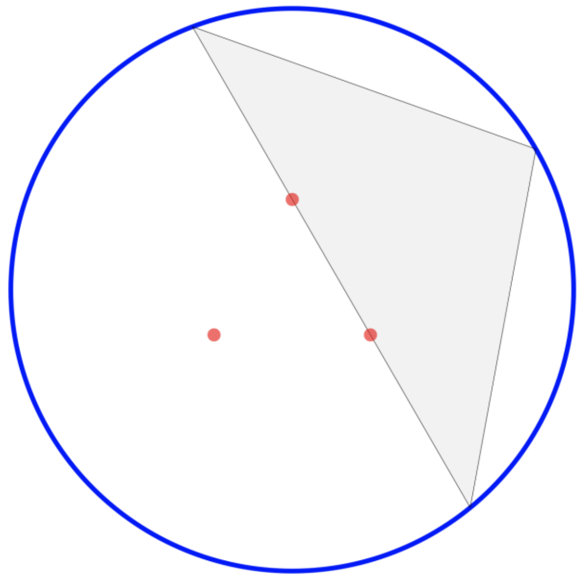
In the interest of simplicity, and because I couldn't think of anything better, I decided to assume that the dots must be placed symmetrically. It seems like a reasonable assumption anyway.

When the three red dots are placed symmetrically, here are some of the most promising triangles that fit among them:



That looks like a lot of triangles but because of the symmetry there are really only two different shapes. They are:





Look what happens when we move the red dots farther apart or closer together: Triangle A gets larger as the dots move apart and smaller as the dots move together. Triangle B does the opposite, getting smaller as the dots move apart and larger as the dots move together.

What we need to do is figure out the Goldilocks case where the triangles are the same size.

Answer

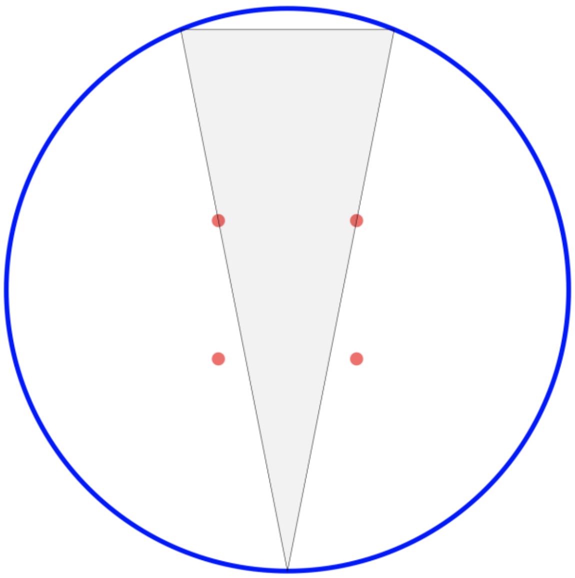
I found that by placing the red dots each 0.320964 away from the center of the disc, triangle A and triangle B both have an area of **0.828637**.

Extra Credit

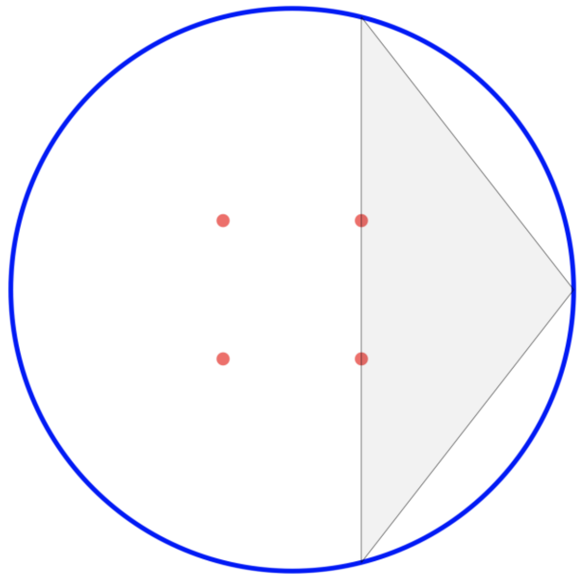
> Instead of placing three points, suppose you're placing *four* points within the unit disk to minimize the area of the largest triangle (by area) that doesn't strictly contain any of the points.
>
> Once again, what is the *minimum* possible area this largest triangle can have?

Again, I'm going to assume that the best placement of the four red dots is symmetrical. I can't think of an asymmetric way of placing the dots that would be better.

With four symmetric dots, the two most promising triangle shapes that fit among the dots are illustrated below:



Triangle Shape A



Triangle Shape B

Triangle A grows in area as the red dots move *farther apart*; triangle B grows in area as the dots move *closer together*. At what point are the triangles equal?

Answer

Placing the red dots 0.347562 units away from the center of the disc makes the two triangles equal in size. Both triangles have an area of **0.7311034**.