Subdividing Squares into Squares

Extra Credit

Suppose you have an infinite supply of square tiles for *each* odd whole number side length (as measured in centimeters) greater than 1 cm. In other words, you have infinitely many 3-by-3 cm tiles, infinitely many 5-by-5 cm tiles, infinitely many 7-by-7 cm tiles, and so on.

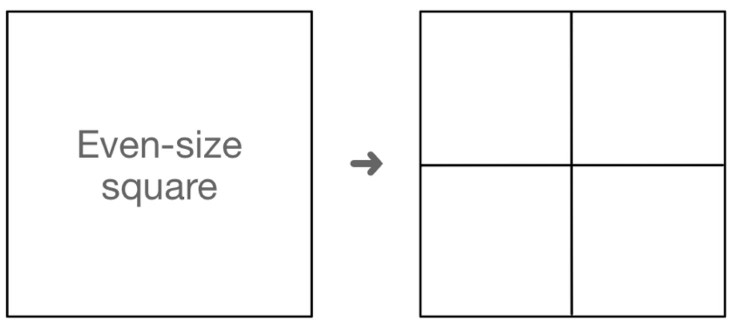
You want to use one or more of these tiles to precisely cover a square whose side length is *N* cm, where *N* is an integer. The tiles may not overlap, and they must completely cover the larger square without jutting beyond its borders. (Before you ask—yes, all odd values of *N* result in squares that can be covered with a single tile.)

What is the *largest* integer *N* for which this task is *not* possible?

There is a pretty straightforward algorithm for covering (just about) any size square:

- If the target square has an *odd* width, use a tile that is the same size as the square. This works for any N except 1, since there is no 1-by-1 tile.

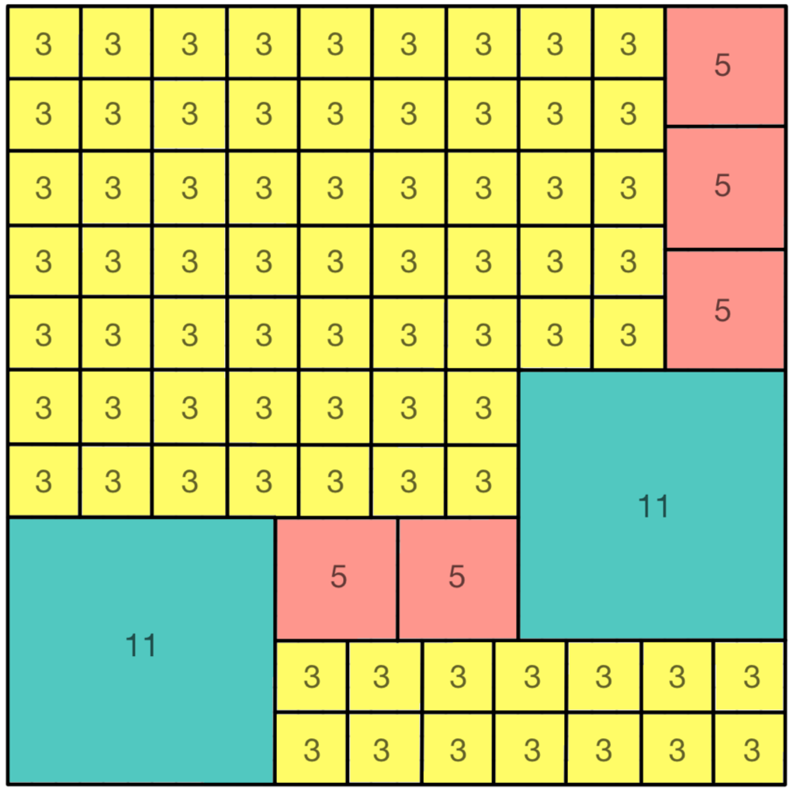
- If the target square has an *even* width, divide it into four equal-sized smaller squares.



  We can now proceed recursively: If the smaller squares have odd width, we are done. If they have even width, subdivide those squares (and, if necessary, subdivide them again) until the subdivided squares have odd width.

This algorithm works on any size square *except* squares that are a power of 2. Squares that are a power of 2 always reduce to 1-by-1 squares and 1-by-1 squares are too small to be covered by the available tiles.

In fact, it turns out to be impossible to tile any square of width 1, 2, 4, 8, or 16. (I know this because I did a computer search.) However, the computer *did* find a solution for 32-by-32:



Since 32-by-32 can be covered in tiles, all the larger powers of 2 (64, 128, etc.) can also be covered in tiles by dividing them into 32-by-32 squares.

The largest square that cannot be covered in tiles is therefore 16-by-16. So N = 16 is the answer to the puzzle.

Disallow 3-by-3 tile?

What if we don't allow 3-by-3 tiles to be used in the solution?

Now the problem sizes are squares that have a width whose prime factors consist of only 2's and 3's. That includes 2, 3, 6, 8, 9, 12, 16, 18, 24, 27, 32, 36, 48, 54, 64, 72, 81, and 96.

I found solutions for widths 54, 64, 72, 81, and 96. These are enough to guarantee that every square larger than 48 can be covered in odd-width tiles greater than 3.

So 48 is the answer for this version of the problem.

Solution for the 96-by-96 square that doesn't use 3-by-3 tiles