Super Bowl Overtime

This week's Fiddler on the Proof (23 February 2024) asks:

This week, it's *your* turn to play football coach. Football is complicated, so let's assume a simplified scoring model.

Every time your team is on offense, suppose there's a 1-in-3 chance they score a touchdown (which we'll say is worth a total of 7 points), a 1-in-3 chance they score a field goal (worth 3 points), and a 1-in-3 chance they don't score any points. After any of these three things happens, your team will then be on defense.

Now, here's how overtime will work: Your team is on offense first. No matter how many points your team does or does not score, the other team then gets a chance at offense. If the game is still tied beyond this point, the teams will continue alternating between offense and defense. Whichever team scores next wins immediately.

Again, your team is on offense first. What is your team's probability of winning?

There are two stages of overtime: **initial overtime**, where each team gets a chance to be on offense, followed if necessary by **sudden death**.

Sudden Death

Since earlier probabilities depend on later probablilities we will look at sudden death first.

Let E be your team's expectation when playing offense in sudden death. If you score a touchdown or field goal, you win the game immediately. If you fail to score, the other team is now on offense. That leads to the following equation for E:

$$E = \frac{2}{3} + \frac{1}{3} \cdot (1 - E)$$

Solving for E gives $3/4$. On offense you win sudden death $3/4$ of the time; on defence you win $1/4$ of the time.

Initial Overtime

In initial overtime, both sides get a chance to play offense. There are 9 possible results each having equal likelihood:

0/0	0/3	0/7
3/0	3/3	3/7
7/0	7/3	7/7

Let's call the first team to play offense **Team A** and the second team to play offense **Team B**. Team A wins initial overtime $1/3$ of the time (3 green squares out of 9). They lose initial

overtime 1/3 of the time (3 red squares out of 9). And the game goes into sudden death with Team A on offense the remaining 1/3 of the time. Team A's expectation is 1/3 + E/3, where E was calculated above to be 3/4. So A's expectation is

$$\frac{1}{3}+\frac{3/4}{3}=\frac{7}{12}$$

Team A's (*your* team's) probability of winning in this scenario is **7/12**.

Extra Credit

> Let's add a smidge more complexity to our model.
>
> If your team happens to score a touchdown on its first possession, then it doesn't make sense for your opponent to then attempt a field goal, since they'd be guaranteed to lose. Instead, they would attempt to score a tying touchdown.
>
> So let's add the following to our model: When either team is on offense, they now have a choice. They can still opt for a strategy that results in 7 points, 3 points, or 0 points, each with a 1-in-3 chance. Alternatively, they can opt for a more aggressive strategy that results in 7 points or 0 points, each with a 1-in-2 chance.
>
> Your team remains on offense first. Assuming both teams play to maximize their own chances of Super Bowl victory, now what is your team's probability of winning?

For convenience, we refer to the aggressive strategy as the "2-way" strategy. We'll refer to the more moderate strategy as the "3-way" strategy.

Sudden Death

In sudden death, the team on offense will adopt the 3-way strategy since that maximizes their chance of scoring. So the situation in sudden death is the same as it was before: the probability of winning if you are on offense is 3/4; the probability of winning if you are on defence is 1/4.

Team B's Strategy

Down 0 points

When the game is tied, B wants to score any way possible, so it will use the 3-way strategy. Two-thirds of the time they win the game. The remaining one third of the time they will go into sudden death playing defence. So Team B's probability of winning is

$$\frac{2+1/4}{3}=\frac{3}{4}$$

Down 7 points

When down 7 points, Team B has to score 7 points to have any chance at all of winning the game, so it will uitilize the 2-way strategy. Half of the time they will lose immediately and the other half of the time the game goes into sudden death with Team B playing defense. Team B's probability of winning is

$$\left(\frac{1}{2}\right)\cdot\left(\frac{1}{4}\right)=\frac{1}{8}$$

Down 3 points

Down 3 points is a little trickier because it is not clear whether Team B wants to employ the 2-way or the 3-way strategy. Let's look at both.

With the 2-way strategy, Team B wins the game half the time, and loses the game the other half of the time, for a probability of winning of $1/2$.

With a 3-way strategy, Team B wins the game $1/3$ of the time, loses the game $1/3$ of the time, and goes into sudden death on defence $1/3$ of the time. Their probability of winning is

$$\frac{1}{3}+\left(\frac{1}{3}\right)\cdot\left(\frac{1}{4}\right)=\frac{5}{12}$$

Since $1/2 > 5/12$, Team B will use the 2-way strategy when it is behind by 3 points.

Table

Points behind	Strategy	Probability of winning
0 points	3-way	3/4
3 points	2-way	1/2
7 points	2-way	1/8

Team A's Strategy

Team A's chances of winning are the complement of Team B's probabilities above.

After scoring	Probability of winning
0 points	1/4
3 points	1/2
7 points	7/8

If Team A uses a 2-way strategy, their probability of winning is the average of the first and third rows of this table.

$$\left(\frac{1}{4}+\frac{7}{8}\right)/2\ =\frac{9}{16}$$

If Team A uses a 3-way strategy, their probability of winning is the average of all 3 rows.

$$\left(\frac{1}{4}+\frac{1}{2}+\frac{7}{8}\right)/3\ =\frac{13}{24}$$

The 2-way strategy is better, so Team A wins the game $\mathbf{9/16}$ of the time.