BACKGAMMON ENDS WITH PROBABILITY ONE

PHILIPPE H. A. CHARMOY

Abstract. In this note I detail a proof that backgammon ends with probability 1, even when the players are clairvoyant, a result due to Curt McMullen in 1994.

1. Introduction

Backgammon is a board game where the players have to move their checkers according to (random) rolls of dice. What we prove here is that the event

$$\{\text{the game ends}\}$$

has probability 1. Define, for $n \in \mathbb{N}$, the event

$$A_n = \{\text{the game ends before the } n\text{-th roll}\}.$$

Then the sequence $(A_n, n \in \mathbb{N})$ is increasing, and

$$\{\text{the game ends}\} = \bigcup_{n \in \mathbb{N}} A_n.$$

And so it is a well-known result of probability theory (see, for example, Theorem 1.1.1 of [1]) that

$$\mathbf{P}(\text{the game ends}) = \lim_{n \to \infty} \mathbf{P}(A_n).$$

And what we prove is that, indeed,

$$\mathbf{P}(A_n) \to 1, \tag{1}$$

as $n \to \infty$. In other words, if you choose $\epsilon > 0$, however small, I can find a positive integer n such that the probability that the game has ended before the n-th roll is $\geq 1 - \epsilon$. It is quite intuitive that this last sentence is equivalent to saying that the event

$$\{\text{the game goes on forever}\}$$

has probability 0. (Recall that this does not mean that it is impossible for the game to go on forever. Suppose that all the rolls are (1,1), then clearly the game can never end. But this happens with probability 0.)

Now obviously, the duration of game depends not only on the result of the rolls but also on the moves chosen by the players. But we will show that (1) holds regardless of the strategy, even if the players are clairvoyant. More explicitly, in a first model, we allow the players to choose their moves according the present rolls and anything that has happened in the past. Then, we show that, given any position of the checkers, there is a finite sequence of rolls that will force the game to end. And we conclude by showing that the players cannot, with positive probability, avoid forever that the sequence about to start is the "right one" given the position of the checkers.

Date: October 3, 2010. Comments and suggestions are gratefully received and should be sent to charmoy@maths.ox.ac.uk.

And in a second model, we allow the player to know the entire sequence of rolls before the game starts. Even in this setup, the players cannot conspire to make the game last forever with positive probability. To prove this, we show that there exists a finite sequence of rolls that will end the game regardless of the current position. And we conclude by showing that, with probability 1, this sequence cannot be avoided forever.

This note should be accessible to anyone having attended a first course in probability. I will address questions of measurability in some footnotes. Readers not interested in these technicalities should feel free to skip them and assume that everything is always well-defined. Also, since excellent sets of rules can be found online,[1] I will not repeat them here; but I will assume that the reader is familiar with them.

My understanding is that the proof that I present here first appeared in 1994 on the online forum rec.games.backgammon,[2] where it was given by Curt McMullen. Now the production of this note has been largely motivated and inspired by an article by Doug Zare, [2], which can also be found online.[3]

For convenience, I included some illustrations. They were produced using Jörg Richter's `bg` package.[4]

2. The model

We will assume, as in chess, that the white start playing and, for definiteness, that the players keep playing forever even when one of them has won — and the game is finished. To be explicit, I will call **white points** the points as counted from the white's perspective and **black points** the points as counted from the black's perspective.

2.1. **Notation.** First, let us agree that, in this document, $\mathbb{N}$ denotes the set of positive integers $\{1, 2, \dots\}$ while $\mathbb{Z}_+$ denotes the set of non-negative integers $\{0, 1 \dots\}$. The n-th roll of dice will be written

$$r_n = r(n) = (r_1(n), r_2(n)) \in E,$$

where $E = \{1, \dots, 6\}^2$ is the set of pairs of numbers between 1 and 6.[5]

We assume that the checkers are numbered — from 1 to 15 for each color. And, for $k, \ell \in \{1, \dots, 15\}$, we write $w_n(k)$, respectively $b_n(\ell)$ for the white point of the k-th white checker after the n-th roll has been played, respectively for the black

[1] See, for example, `http://www.bkgm.com/rules.html`.

[2] See `http://groups.google.com/group/rec.games.backgammon/`, and search for "termination of backgammon".

[3] See, for example, `http://www.math.harvard.edu/~ctm/expositions/bgends.html`.

[4] See `http://www.tex.ac.uk/CTAN/macros/latex/contrib/bg/description.pdf`.

[5] The σ-algebra with which we equip E is its power set. In practice, players typically cannot distinguish between the dice, and so we may prefer to take E to be the set of unordered pair $\{1, \dots, 6\}^2/\sim$, where $\sim$ is the equivalence relation defined by

$$(a, b) \sim (c, d) \iff (a, b) = (c, d) \text{ or } (a, b) = (d, c).$$

This reduces the number of strategies available to the players and does not change measurability questions since we would also equip $\{1, \dots, 6\}^2/\sim$ with its power set. For convenience, I choose the definition of E that makes all singletons equiprobable.

point of the ℓ-th black checker after the n-th roll has been played. Then the position after the n-th roll has been played is determined by

$$p_n = p(n) = (w_1(n), \ldots, w_{15}(n), b_1(n), \ldots, b_{15}(n)) \in G,$$

where $G = \{0, \ldots, 25\}^{15}$ is the set of 15-tuples of numbers between 0 and 25.[6] We will write p_0 for the initial position given by the rules.

When the players are not clairvoyant, there must exist a sequence of functions $(f_n, n \in \mathbb{N})$ such that, for every $n \in \mathbb{N}$,

$$p_n = f_n(p_{n-1}, r_n),$$

that contain the information about the strategies of the players and forces them only to take into account past information.[7]

2.2. **The probability space and random variables.** We need a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ on which to define a sequence $(R_n, n \in \mathbb{N})$ of independent and uniformly distributed E-valued random variables: the rolls.[8]

Now notice that

$$A_n = \{W_1(n) = \cdots = W_{15}(n) = 0 \text{ or } B_1(n) = \cdots = B_{15}(n) = 0\},$$

where a capitalized letter indicates a random variable.[9]

Also, when the players are not clairvoyant, P_n is independent of $R_{n+1}, R_{n+2}, \ldots$, for every $n \in \mathbb{Z}_+$.

3. Forcing the game to end from a given position

In this section, we prove that given any position of the checkers, there is a sequence of rolls that will force the game to end.

We start by showing that the sequence (2,4) can end the game in finitely many rolls if the game does not get stuck before.

Proposition 3.1. *Let p be any position of the checkers on the board. Then, after 9000 rolls equal to (2,4):*

(i) *the game is stuck, i.e. in a situation where neither player can play either the 2 or the 4, and the game is not finished; or*
(ii) *the game is finished.*

Proof. Suppose that the game does not get stuck and define the modified pip count **mpc** by

$$\begin{aligned}\mathbf{mpc}_n = &\sum_{k=1}^{15} \left[W_k(n)\mathbf{1}_{W_k(n) \text{ is odd}} + B_k(n)\mathbf{1}_{B_k(n) \text{ is odd}}\right] \\ &+ 12.5 \sum_{k=1}^{15} \left[W_k(n)\mathbf{1}_{W_k(n) \text{ is even}} + B_k(n)\mathbf{1}_{B_k(n) \text{ is even}}\right].\end{aligned}$$

(Here, $\mathbf{1}_{\text{statement}}$ is the function that equals 1 when the statement is true and 0 otherwise.) That is to say, we weigh the contribution of checkers on even points

[6]Again, the σ-algebra with which we equip G is its power set.

[7]Since, for every n, the domain and codomain of f_n are equipped with their power set, every f_n is measurable.

[8]The existence of such a space is guaranteed by Kolmogorov's extension theorem.

[9]This proves that all the events A_n as well as the event {the game ends} are measurable.

12.5 times more. Notice that if $\mathbf{mpc}_n = 0$ the game must have been ended before the n-th roll.

Now let us show that the **mpc** decreases by at least 2 each time a player moves. Indeed:

- if no hits are made, the modified pip count decreases by at least 2 since at least one player will have moved forward by 2;
- if a hit is made on odd point (of the player whose turn it is), the move decreases the modified pip count by at least 2, and sending the opponent's checker on the bar cannot increase the modified pip count; indeed the checker that was hit was on an even point (of its player) and so contributed at least 25 to the modified pip count before the hit while contributing exactly 25 after the hit;
- if a hit is made on even point (of the player whose turn it is), the move decreases the modified pip count by at least 25, and sending the opponent's checker on the bar cannot increase the modified pip count by more than 23; indeed the checker that was hit was on an odd point (of its player) that is ≥ 2.

Since the game does not get stuck, each exchange includes one move and thus reduces the modified pip count by 2. But also, since a checker contributes most to the modified pip count when it is on point 24, and since there are 30 checkers, the modified pip count is bounded by

$$9000 = 30 * 24 * 12.5.$$

So, after 9000 moves, there will have been 4500 exchanges so that $\mathbf{mpc}_{9000} = 0$, as required. $\square$

The intuition for this proof is a follows. If the game does not get stuck, it cannot cycle, because whenever a checker is hit, it will be placed on an odd point and will remain on a odd point forever. So there is some sort of progression. A similar argument can be made using the sequence (3,6) and gives a shorter sequence. However, it is not as convenient to write down since we cannot use the terminology of odd and even. The details are provided in Section 7.

Our next step is to give a sufficient condition for the sequence above not to get stuck.

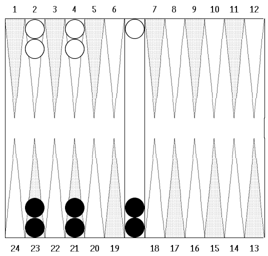
Proposition 3.2. *Let p be any position in which at most one player has checkers on the bar. Then the game does not get stuck along a sequence of (2,4).*

Proof. First, let us prove that either the white or the black can play part of a (2,4) from p. If player can not play, then:

- they have a point, say a, between 5 and 25, such that the opponent has made point 2 and 4 pips ahead (that is on points $a-2$ and $a-4$ of the player whose turn it is); or
- they have a checker on the bar.

But one of the players, say John Smith, has no checkers on the bar. So his opponent, say Jane Doe, can move from her point $29-a$ to her point $27-a$.

Second, notice that, starting from p, even if a situation where both players are on the bar is reached, at least one player can play part of a (2,4). Indeed if both players are on the bar and none can play, then both players must have made their points 23 and 21:



Situation where the sequence of (2,4) gets stuck.

But initially, one of the players, say John, did not have checkers on the bar. So for the game to get stuck, Jane must hit from the bar with a (2,4). Thus, the John cannot have made points 2 and 4. Now if Jane has not made points 2 and 4, then John can remove his checker from the bar at the next roll. Otherwise, he is stuck, but Jane can remove all her checkers by playing the sequence of (2,4). This will again create a situation where at most one player has checkers on the bar.

So at least one player will always be able to play, as required. □

The final element we need is thus a way to get out of a situation in which both players have checkers on the bar.

Proposition 3.3. *Let p be any position. Then there exists a sequence of 8 rolls after which at most one player can have checkers on the bar.*

Proof. It is impossible for both players to be shut off. (One player needs to be shut off first.) Now suppose there is a white point, say a, between 24 and 19 where the black have not made point. Then any sequence of the form

$$(b, b), (-, -), (b, b), (-, -), (b, b), (-, -), (b, b), (-, -)$$

where $b = 25 - a$ and $-$ means "anyting", will do. Indeed after the n-th pair of rolls the white cannot have more than $\max\{15-4n, 0\}$ checkers on the bar. If there is no such point a, then the white cannot play, and there must exist a black point, say c, between 24 and 19 where the white have not made point. Then any sequence of the form

$$(-, -), (d, d), (-, -), (d, d), (-, -), (d, d), (-, -), (d, d),$$

where $d = 25 - c$, will do. □

We now have all we need to prove the result of this section.

Theorem 3.4. *Let p be any position. Then there exists a sequence of 9008 rolls after which the game must have ended.*

Proof. Suppose first that both players have checkers on the bar in position p. Then use Proposition 3.3 to find a sequence of 8 rolls after which at most one players can have checkers on the bar. Then, by Propositions 3.1 and 3.2, after 9000 rolls equal to (2,4), the game must have ended.

Otherwise, if at most one players has checkers on the bar in position p, then, by Propositions 3.1 and 3.2, after 9008 rolls equal to (2,4), the game must have ended.

The proof is complete. □

4. Backgammon ends when the players are not clairvoyant

In this section, we prove that, with probability 1, backgammon ends. To the idea is to show that the players cannot, with positive probability, avoid forever the sequences that force the game to end.

Theorem 4.1. *When the players are not clairvoyant, backgammon ends with probability 1.*

This proof is essentially a calculation that uses the independence of the rolls.

Proof. Let M be a number such that, from any position p, there exists a sequence $S(p)$ of rolls of length M after which the game must be finished. (Theorem 3.4 shows that 9008 is such a number.) And define, for all $n \in \mathbb{N}$ and $p \in G$,

$$B_{m,n}(p) = \{\text{the game starting from position } p \text{ at the } (m+1)\text{-th roll has ended before the } (m+n)\text{-th roll}\}.$$

(Notice in particular that $A_n = B_{0,n}(p_0)$.)

We show by induction that, for all $k \in \mathbb{N}$ and $p \in G$,

$$\mathbf{P}(B_{0,kM}(p)^c) \leq (1 - 36^{-M})^k.$$

For $k = 1$, we have that

$$\mathbf{P}(B_{0,M}(p)^c) \leq 1 - \mathbf{P}((R_1, \ldots, R_M) = S(p)) = 1 - 36^{-M}.$$

For k arbitrary, and assuming that the result is true for $k-1$, we have

$$\begin{aligned}
\mathbf{P}(B_{0,kM}(p)) &= \mathbf{P}(B_{0,M}(p)^c, B_{M,(k-1)M}(P_M)^c) \\
&= \sum_{p_M \in G} \mathbf{P}(B_{0,M}(p)^c, B_{M,(k-1)M}(P_M)^c, P_M = p_M) \\
&= \sum_{p_M \in G} \mathbf{P}(B_{0,M}(p)^c, B_{M,(k-1)M}(p_M)^c, P_M = p_M) \\
&= \sum_{p_M \in G} \mathbf{P}(B_{0,M}(p)^c, P_M = p_M)\mathbf{P}(B_{M,(k-1)M}(p_M)^c) \\
&= \sum_{p_M \in G} \mathbf{P}(B_{0,M}(p)^c, P_M = p_M)\mathbf{P}(B_{0,(k-1)M}(p_M)^c),
\end{aligned}$$

where we used that $B_{0,M}$ and P_M are independent of $B_{M,(k-1)M}$ and that the the rolls are identically distributed. Now, by the induction hypothesis, we get

$$\begin{aligned}
\mathbf{P}(B_{0,kM}(p)) &\leq (1 - 36^{-M})^{k-1} \sum_{p_M \in G} \mathbf{P}(B_{0,M}(p)^c, P_M = p_M) \\
&= (1 - 36^{-M})^{k-1}\mathbf{P}(B_M(p)^c) \\
&\leq (1 - 36^{-M})^k,
\end{aligned}$$

which is the desired inequality.

To conclude, notice that

$$\mathbf{P}(A^c_{kM}) = \mathbf{P}(^B_{kM}(p_0)^c) \leq (1 - 36^{-M})^k.$$

Letting $k \to \infty$ show that

$$\mathbf{P}(A^c) = 0,$$

as required. $\square$

The proof crucially uses that B_M and P_M are independent of $B_{(k-1)M}(p_M)$. Since this is no longer true when we allow the players to be clairvoyant, the proof no longer works in that case. But we can fix that if there is a strategy that forces the game to end, regardless of the initial position.

5. Forcing the game to end regardless of the position

The purpose of this section is to build a finite sequence of rolls that forces the game to end regardless of the initial position. The sequence built here is essentially a concatenation of sequences suggested by the construction of Section 3. This is the only result of this section.

Theorem 5.1. *There exists a sequence of rolls of length 81064 after which the game must be finished, regardless of the initial position of the checkers.*

Proof. Let p be any position of the checkers. Denote by S the sequence of $(2,4)$ of length 9000. Now, for $i \in \{1,3,5,6\}$ define the sequences

$$T_i = (i,i),(2,4),(i,i),(2,4),(i,i),(2,4),(i,i),(2,4)$$
$$U_i = (2,4),(i,i),(2,4),(i,i),(2,4),(i,i),(2,4),(i,i),$$

and set

$$V = S, T_1, S, T_3, S, T_5, S, T_6, S, U_1, S, U_3, S, U_5, S, U_6, S.$$

We show that V forces the game to finish. Suppose that the game is not finished after the first S. By Proposition 3.1, the game is stuck in a situation where both players have checkers on the bar and have made their points 23 and 21. Now suppose their is a white point between 24 and 19 that is not made by the black. Eventually, the sequence

$$T_1, S, T_3, S, T_5, S, T_6, S$$

will remove all white checkers from the bar without moving any black checker and will force the game to end. Now if the white are shut off, then black cannot be, and, with a similar argument, the sequence

$$U_1, S, U_3, S, U_5, S, U_6, S$$

will force the game to end.

Finally, notice that V has length

$$81064 = 9 * 9000 + 8 * 8,$$

as required. $\square$

6. Backgammon ends, even when the players are clairvoyant

We can now prove that backgammon ends in the strongest version.

Theorem 6.1. *Backgammon ends, with probability 1, even if the players are clairvoyant.*

Proof. Let $N = 81064$, and notice that, for every $k \in \mathbb{N}$,

$$\begin{aligned}
\mathbf{P}(A_{kN}^c) &\leq \mathbf{P}((R_1,\ldots,R_N) \neq V, \ldots, (R_{(k-1)N+1},\ldots,R_{kN}) \neq V) \\
&= \mathbf{P}((R_1,\ldots,R_N) \neq V) \cdots \mathbf{P}((R_{(k-1)N+1},\ldots,R_{kN}) \neq V) \\
&= [1 - \mathbf{P}((R_1,\ldots,R_N) = V)] \cdots [1 - \mathbf{P}((R_{(k-1)N+1},\ldots,R_{kN}) \neq V)] \\
&= [1 - 36^{-N}]^k.
\end{aligned}$$

Letting $k \to \infty$ yields the result. □

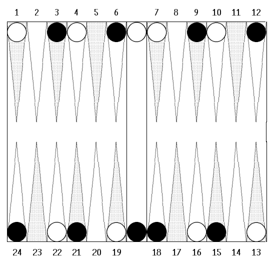
7. Bounds for the length of sequences that force the game to end

In this section, we show how the use of a sequence of $(3,6)$ improves the bound of Proposition 3.1.

Proposition 7.1. *Let p be any position of the checkers on the board. Then, after 6008 rolls equal to (3,6):*

(i) *the game is stuck, i.e. in a situation where neither player can play either the 3 or the 6, and the game is not finished; or*
(ii) *the game is finished.*

Proof. Say that a point is **right** if it is in $25 - 3\mathbb{Z}$ and say that it is **wrong** otherwise. Notice that the right points for the white and the black are disjoint sets:



White and black on their right points, including the bar.

Suppose that the game does not get stuck and define the modified pip count **mpc** by

$$\begin{aligned}
\mathbf{mpc}_n = &\sum_{k=1}^{15} \left[W_k(n)\mathbf{1}_{W_k(n) \text{ is right}} + B_k(n)\mathbf{1}_{B_k(n) \text{ is right}}\right] \\
&+ 12.5 \sum_{k=1}^{15} \left[W_k(n)\mathbf{1}_{W_k(n) \text{ is wrong}} + B_k(n)\mathbf{1}_{B_k(n) \text{ is wrong}}\right].
\end{aligned}$$

Now let us show that the **mpc** decreases by at least 3 each time a player moves. Indeed, arguing as in Proposition 3.1:

- if not hits are made, the modified pip count decreases by at least 3;
- if right hits wrong, the modified pip count decreases by at least 3 because any checker on a wrong point contributes at least 25 to the modified pip count;

- if wrong hits right, the modified pip count decreases by at least 3 because the move from wrong to wrong reduces the modified pip count by 37.5 while the hit increases it by no more than 24;
- if wrong hits wrong, the modified pip count decreases by at least 3 because the move from wrong to wrong reduces the modified pip count by 37.5 while the hit cannot increase the modified pip count since any checker on a wrong point contributes at least 25 to the modified pip count.

Since the game does not get stuck, each exchange reduces the modified pip count by 3. But also, since a checker contributes most when it is on point 24, as in Proposition 3.1, the modified pip count is bounded by 9000. So, after 6000 moves, there will have been 3000 exchanges so that $\mathbf{mpc}_{6000} = 0$ and the game must be finished, as required. □

Using this, we easily get that, from any position, there is a sequence of 6008 rolls that ends the game. And there is a sequence of 54064 rolls that ends the game regardless of the position.

8. Concluding remarks

In this note, we have shown that there are finite sequences of rolls that force the game to end. Moreover, we have proved that there exists such a sequence that will work from any position of the checkers. We have then used the results to prove that backgammon ends with probability 1, even when the players are clairvoyant.

Most likely, there is room for improvement for the length of the sequence of (3,6) in Proposition 7.1. Maybe a better bound can be found with a combinatorial argument, or, more brutally, by a direct simulation.

9. Acknowledgments

I am grateful to Dr. Doug Zare for making his article about the proof available online and agreeing that I put a link to this note on my web page. I must also thank Prof. Curt McMullen for a few emails explaining how to treat the case of clairvoyant players.

References

[1] R.T. Durrett. *Probability: Theory and Examples, Fourth Edition.* Cambridge University Press, 2010.

[2] D. Zare. *Backgammon Ends.* GammonVillage, 2000.