Julien and Kyle Flip Coins

This week's Fiddler on the Proof (15 December 2023) asks:

> Kyle and Julien are playing a game in which they each toss their own fair coins. On each turn of the game, both players flip their own coin once.
>
> If, at any point, Kyle's most recent three flips are Tails, Tails, and Heads (i.e., TTH), then he wins. If, at any point, Julien's most recent three flips are Tails, Tails, and Tails (i.e, TTT), then *he* wins.
>
> However, both players can't win at the same time. If Kyle gets TTH at the same time Julien gets TTT, then *no one* wins, and they continue flipping. They don't start over completely or erase their history, mind you—they merely continue flipping, so that one of them could conceivably win in the next flip or two.
>
> What is the probability that Kyle wins this game?

Let's refer to the players' two previous flips as their "coin histories." Taking both players together, there are 16 possible coin histories. We will call this the "state" of the game.

From each state, 4 possible results can happen: both players flip tails (T/T), Julien flips tails and Kyle flips heads (T/H), Julien flips heads and Kyle flips tails (H/T), or they both flip heads (H/H).

The Kyle's probability of winning from each state can be computed as laid out in the following table:

Current State: Julien's history / Kyle's history	Kyle's probability of winning from here	New state after players' flips Julien's flip / Kyle's flip T / T	T / H	H / T	H / H
TT / TT	0.577827	**Julien wins**	TT / TH	TH / TT	**Kyle Wins**
TT / TH	0.349689	TT / HT	TT / HH	TH / HT	TH / HH
TT / HT	0.400965	TT / TT	TT / TH	TH / TT	TH / TH
TT / HH	0.349689	TT / HT	TT / HH	TH / HT	TH / HH
TH / TT	0.961621	HT / TT	HT / TH	HH / TT	HH / TH
TH / TH	0.642238	HT / HT	HT / HH	HH / HT	HH / HH
TH / HT	0.756518	HT / TT	HT / TH	HH / TT	HH / TH
TH / HH	0.642238	HT / HT	HT / HH	HH / HT	HH / HH
HT / TT	0.884862	TT / TT	TT / TH	TH / TT	TH / TH
HT / TH	0.537353	TT / HT	TT / HH	TH / HT	TH / HH
HT / HT	0.632844	TT / TT	TT / TH	TH / TT	TH / TH
HT / HH	0.537353	TT / HT	TT / HH	TH / HT	TH / HH
HH / TT	0.961621	HT / TT	HT / TH	HH / TT	HH / TH
HH / TH	0.642238	HT / HT	HT / HH	HH / HT	HH / HH
HH / HT	0.756518	HT / TT	HT / TH	HH / TT	HH / TH
HH / HH	0.642238	HT / HT	HT / HH	HH / HT	HH / HH
Average	**0.642238**				

Kyle's probability of winning from each state (each row of the table) is the average of the probabilities of the 4 possible new states.

His probability of winning overall is the average of his probabilities of the 16 states, which works out to be about 64.2238%.

Extra Credit

Kyle and Julien write down all eight possible sequences for three coin flips (HHH, HHT, HTH, HTT, THH, THT, TTH, and TTT) on eight different slips of paper. They place these slips into a hat and shake it.

They will each randomly draw slips of paper out of the hat, at which point they will play the same game as previously described, but looking for the sequence specified on the slip of paper they each selected. Kyle draws first and looks at his slip of paper. After doing some calculations, he says: "Well, at this point, it's about as fair a match as it could possible be."

Which slip of paper might Kyle have drawn? And what are his chances of winning at this point (i.e., before Julien selects his own slip of paper)?

Kyle's Probability of Winning after Each Possible Pick

Kyle's pick	Julien's Pick								
	TTT	**TTH**	**THT**	**THH**	**HTT**	**HTH**	**HHT**	**HHH**	**Average**
TTT		.357762	.412982	.357762	.357762	.412982	.357762	.500000	**.393859**
TTH	.642238		.549730	.500000	.500000	.549730	.500000	.642238	**.554848**
THT	.587018	.450270		.450270	.450270	.500000	.450270	.587018	**.496445**
THH	.642238	.500000	.549730		.500000	.549730	.500000	.642238	**.554848**
HTT	.642238	.500000	.549730	.500000		.549730	.500000	.642238	**.554848**
HTH	.587018	.450270	.500000	.450270	.450270		.450270	.587018	**.496445**
HHT	.642238	.500000	.549730	.500000	.500000	.549730		.642238	**.554848**
HHH	.500000	.357762	.412982	.357762	.357762	.412982	.357762		**.393859**

If Kyle picks the slip containing THT, his average probability of winning against each of Julien's 7 possible picks is 0.496445. That's as close as possible to a fair match. Likewise with HTH.