Summing Cards

This week's Fiddler on the Proof (16 February 2024) asks:

> You and a friend have a large deck of cards, all of which are numbered 1, 2, 3, or 4. There are many of each of these numbers in the deck.
>
> You alternate placing down one card at a time in a pile. If, at any point, the sum of the most recently played group of cards equals the sum of a group of cards played immediately before them, then you and your friend both slap the pile. Whoever slaps first wins the pile.
>
> Here are some sequences of cards that would be slapped once the last card in the sequence is played:
>
> - 3, 2, 3, 4, 1 (because the last two cards have a sum equal to that of the two cards prior)
> - 1, 2, 4, 3, 3 (because the last one card has a "sum" equal to that of the one card prior)
> - 2, 3, 1, 2 (because the last two cards have a sum equal to that of the one card prior)
>
> How many cards are in the longest possible sequence that is *never* slapped?
>
> **Extra Credit**
>
> In the preceding puzzle, the numbers on the cards were 1 through 4. Suppose, instead, they were numbered 1 through *N*.
>
> When *N* is 5, how many cards are in the longest possible sequence that is never slapped? What if *N* is 6? What if *N* is 7?

Answer

I wrote a computer program to look for the longest sequences of each N from 1 to 9. Here are the results.

N	Number of maximal sequences	Length of longest sequence
1	1	1
2	2	3
3	8	7
4	98	9
5	1,508	15
6	81,151	21
7	3,188,153	31
8	324,402,815	35
9	69,107,820,767	47

A **maximal sequence** is a sequence that cannot be extended by adding another number onto the end. That's because any number you might add would create a repeated sum. For example, here are the 8 maximal sequences for N = 3:

1, 2, 1	2, 3, 1, 3, 2, 3, 1
1, 3, 1	2, 3, 2
1, 3, 2, 3, 1, 3, 2	3, 1, 3, 2, 3, 1, 3
2, 1, 2	3, 2, 3, 1, 3, 2, 3

Notice that the reverse of a maximal sequence is also a maximal sequence.

Question: Why are the longest lengths all odd numbers? Or is it just a coincidence?

Partition Plots

Looking at the following plots, it is fairly easy to see that no adjacent sums are repeated in the illustrated sequences of cards.

The top row of each chart is the sequence. The rows beneath it partition the sequence at different points. The blue numbers add up the cards going backwards from the partition. The red numbers add up the cards going forward from the partition.

The thing to notice is that the numbers in each row are all different. In particular, each of the blue sums is different to each of the red sums.

4 values, sequence of length 9:

1	4	3	4	2	4	3	4	1
1 |	4	7	11	13	17	20	24	25
5	4 |	3	7	9	13	16	20	21
8	7	3 |	4	6	10	13	17	18
12	11	7	4 |	2	6	9	13	14
14	13	9	6	2 |	4	7	11	12
18	17	13	10	6	4 |	3	7	8
21	20	16	13	9	7	3 |	4	5
25	24	20	17	13	11	7	4 |	1

5 values, sequence of length 15:

1	3	2	4	5	3	4	5	1	3	2	4	5	3	4
1 |	3	5	9	14	17	21	26	27	30	32	36	41	44	48
4	3 |	2	6	11	14	18	23	24	27	29	33	38	41	45
6	5	2 |	4	9	12	16	21	22	25	27	31	36	39	43
10	9	6	4 |	5	8	12	17	18	21	23	27	32	35	39
15	14	11	9	5 |	3	7	12	13	16	18	22	27	30	34
18	17	14	12	8	3 |	4	9	10	13	15	19	24	27	31
22	21	18	16	12	7	4 |	5	6	9	11	15	20	23	27
27	26	23	21	17	12	9	5 |	1	4	6	10	15	18	22
28	27	24	22	18	13	10	6	1 |	3	5	9	14	17	21
31	30	27	25	21	16	13	9	4	3 |	2	6	11	14	18
33	32	29	27	23	18	15	11	6	5	2 |	4	9	12	16
37	36	33	31	27	22	19	15	10	9	6	4 |	5	8	12
42	41	38	36	32	27	24	20	15	14	11	9	5 |	3	7
45	44	41	39	35	30	27	23	18	17	14	12	8	3 |	4

6 values, sequence of length 21:

3	5	4	6	4	3	5	4	6	5	6	2	6	5	6	1	3	5	6	1	6
3	5	9	15	19	22	27	31	37	42	48	50	56	61	67	68	71	76	82	83	89
8	5	4	10	14	17	22	26	32	37	43	45	51	56	62	63	66	71	77	78	84
12	9	4	6	10	13	18	22	28	33	39	41	47	52	58	59	62	67	73	74	80
18	15	10	6	4	7	12	16	22	27	33	35	41	46	52	53	56	61	67	68	74
22	19	14	10	4	3	8	12	18	23	29	31	37	42	48	49	52	57	63	64	70
25	22	17	13	7	3	5	9	15	20	26	28	34	39	45	46	49	54	60	61	67
30	27	22	18	12	8	5	4	10	15	21	23	29	34	40	41	44	49	55	56	62
34	31	26	22	16	12	9	4	6	11	17	19	25	30	36	37	40	45	51	52	58
40	37	32	28	22	18	15	10	6	5	11	13	19	24	30	31	34	39	45	46	52
45	42	37	33	27	23	20	15	11	5	6	8	14	19	25	26	29	34	40	41	47
51	48	43	39	33	29	26	21	17	11	6	2	8	13	19	20	23	28	34	35	41
53	50	45	41	35	31	28	23	19	13	8	2	6	11	17	18	21	26	32	33	39
59	56	51	47	41	37	34	29	25	19	14	8	6	5	11	12	15	20	26	27	33
64	61	56	52	46	42	39	34	30	24	19	13	11	5	6	7	10	15	21	22	28
70	67	62	58	52	48	45	40	36	30	25	19	17	11	6	1	4	9	15	16	22
71	68	63	59	53	49	46	41	37	31	26	20	18	12	7	1	3	8	14	15	21
74	71	66	62	56	52	49	44	40	34	29	23	21	15	10	4	3	5	11	12	18
79	76	71	67	61	57	54	49	45	39	34	28	26	20	15	9	8	5	6	7	13
85	82	77	73	67	63	60	55	51	45	40	34	32	26	21	15	14	11	6	1	7
86	83	78	74	68	64	61	56	52	46	41	35	33	27	22	16	15	12	7	1	6

7 values, sequence of length 31:

1	7	6	7	5	7	6	4	3	4	6	7	5	7	6	7	1	7	6	7	5	7	6	4	3	4	6	7	5	7	6
1	7	13	20	25	32	38	42	45	49	55	62	67	74	80	87	88	95	101	108	113	120	126	130	133	137	143	150	155	162	168
8	7	6	13	18	25	31	35	38	42	48	55	60	67	73	80	81	88	94	101	106	113	119	123	126	130	136	143	148	155	161
14	13	6	7	12	19	25	29	32	36	42	49	54	61	67	74	75	82	88	95	100	107	113	117	120	124	130	137	142	149	155
21	20	13	7	5	12	18	22	25	29	35	42	47	54	60	67	68	75	81	88	93	100	106	110	113	117	123	130	135	142	148
26	25	18	12	5	7	13	17	20	24	30	37	42	49	55	62	63	70	76	83	88	95	101	105	108	112	118	125	130	137	143
33	32	25	19	12	7	6	10	13	17	23	30	35	42	48	55	56	63	69	76	81	88	94	98	101	105	111	118	123	130	136
39	38	31	25	18	13	6	4	7	11	17	24	29	36	42	49	50	57	63	70	75	82	88	92	95	99	105	112	117	124	130
43	42	35	29	22	17	10	4	3	7	13	20	25	32	38	45	46	53	59	66	71	78	84	88	91	95	101	108	113	120	126
46	45	38	32	25	20	13	7	3	4	10	17	22	29	35	42	43	50	56	63	68	75	81	85	88	92	98	105	110	117	123
50	49	42	36	29	24	17	11	7	4	6	13	18	25	31	38	39	46	52	59	64	71	77	81	84	88	94	101	106	113	119
56	55	48	42	35	30	23	17	13	10	6	7	12	19	25	32	33	40	46	53	58	65	71	75	78	82	88	95	100	107	113
63	62	55	49	42	37	30	24	20	17	13	7	5	12	18	25	26	33	39	46	51	58	64	68	71	75	81	88	93	100	106
68	67	60	54	47	42	35	29	25	22	18	12	5	7	13	20	21	28	34	41	46	53	59	63	66	70	76	83	88	95	101
75	74	67	61	54	49	42	36	32	29	25	19	12	7	6	13	14	21	27	34	39	46	52	56	59	63	69	76	81	88	94
81	80	73	67	60	55	48	42	38	35	31	25	18	13	6	7	8	15	21	28	33	40	46	50	53	57	63	70	75	82	88
88	87	80	74	67	62	55	49	45	42	38	32	25	20	13	7	1	8	14	21	26	33	39	43	46	50	56	63	68	75	81
89	88	81	75	68	63	56	50	46	43	39	33	26	21	14	8	1	7	13	20	25	32	38	42	45	49	55	62	67	74	80
96	95	88	82	75	70	63	57	53	50	46	40	33	28	21	15	8	7	6	13	18	25	31	35	38	42	48	55	60	67	73
102	101	94	88	81	76	69	63	59	56	52	46	39	34	27	21	14	13	6	7	12	19	25	29	32	36	42	49	54	61	67
109	108	101	95	88	83	76	70	66	63	59	53	46	41	34	28	21	20	13	7	5	12	18	22	25	29	35	42	47	54	60
114	113	106	100	93	88	81	75	71	68	64	58	51	46	39	33	26	25	18	12	5	7	13	17	20	24	30	37	42	49	55
121	120	113	107	100	95	88	82	78	75	71	65	58	53	46	40	33	32	25	19	12	7	6	10	13	17	23	30	35	42	48
127	126	119	113	106	101	94	88	84	81	77	71	64	59	52	46	39	38	31	25	18	13	6	4	7	11	17	24	29	36	42
131	130	123	117	110	105	98	92	88	85	81	75	68	63	56	50	43	42	35	29	22	17	10	4	3	7	13	20	25	32	38
134	133	126	120	113	108	101	95	91	88	84	78	71	66	59	53	46	45	38	32	25	20	13	7	3	4	10	17	22	29	35
138	137	130	124	117	112	105	99	95	92	88	82	75	70	63	57	50	49	42	36	29	24	17	11	7	4	6	13	18	25	31
144	143	136	130	123	118	111	105	101	98	94	88	81	76	69	63	56	55	48	42	35	30	23	17	13	10	6	7	12	19	25
151	150	143	137	130	125	118	112	108	105	101	95	88	83	76	70	63	62	55	49	42	37	30	24	20	17	13	7	5	12	18
156	155	148	142	135	130	123	117	113	110	106	100	93	88	81	75	68	67	60	54	47	42	35	29	25	22	18	12	5	7	13
163	162	155	149	142	137	130	124	120	117	113	107	100	95	88	82	75	74	67	61	54	49	42	36	32	29	25	19	12	7	6