PROPER RAISING POINTS IN A GENERALIZATION OF BACKGAMMON

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SUMMARY

In two person perfect information games with chance, the players theoretically can know their probability of winning, p(t). In the game treated herein, p(t) is assumed to be continuous. As the game progresses, players can bet, following the fixed rules for betting, that they will win. Optimal strategies are derived for the original backgammon betting rules, and for a variety of other betting rules.

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1. INTRODUCTION

Backgammon is a perfect information game with both players' moves governed by alternating throws of dice. Thus, in theory, there is a best strategy for each player to maximize his chance of winning, and, at any time, each player can compute his exact probability of winning if the best strategies are followed. Players also are allowed to raise the stakes if they obey the betting rules, which will be described in the next section. Though it may upset Backgammon aficionados, Gammons and Backgammons will only be considered as simple wins. We will describe the optimal betting strategy for a continuous version of the game.

We let time t be the variable. Let $p(t)$ = the probability that player A will win at time t (given both players play optimally). It will be assumed throughout that $p(t)$ is a continuous function of time (and say the game is continuous). This is not true of any discrete game, where single moves may cause $p(t)$ to jump [and sometimes drastically, as in Monopoly, when Boardwalk with a hotel is hit, Sorry, when a Sorry Card is played, and Backgammon, especially in the end game when doubles occur]. Being mathematicians, however, we ignore such practicalities, and assume our game "continuous."

Given a continuous game, consider the associated game [called the P-game] consisting of the unit interval [0,1] and an indicator. This indicator is at point $p(t)$ [which we'll call the state of the P—game] at time t. Player A wins if the indicator reaches 1 before it reaches 0. We assume throughout that, with probability one, $p(t) = 0$ or 1 for some finite t. It will turn out that, for the different doubling rules we consider, the state of this P—game will determine the strategy.

<u>LEMMA</u>: *Let the P—game be at state x, with a, b > 0 such that $0 \leq x-a < x < x+b \leq 1$. Let E be the event that p(t) becomes x+b before it becomes x—a. Then the probability of E is a/a+b.*

<u>PROOF</u>: The state x = Prob [A will win] = P(E)P[A will win|E] + P(~E)P[A will win | ~E]. But since p(t) is Markovian, P[A will win | p(t) becomes x+b before x—a] = P[A will win | p(0) = x+b] = x+b. Similarly P[A will win | ~E] = x—a. So

$$x = P(E)(x+b) + (1 - P(E))(x-a) \tag{1}$$

and therefore

$$P(E) = \frac{a}{a+b} \tag{2}$$

This lemma shows that our P—game indicator moves as though in a random walk on [0,1] but possibly on a different time scale. Thus, in this respect, all continuous games are equivalent to a random walk game, where an indicator moves as Brownian motion on [0,1] and player A wins if the indicator hits 1 before it hits 0.

2. STANDARD BACKGAMMON BETTING

In the game of Backgammon, a "doubling die" is used to record the stakes. At the beginning of the game, the die is in the center with "1" face up. If this die is never touched throughout the game, the loser pays the winner one unit. At any time in the game either player can double the stakes, whereupon his opponent can quit and pay the doubler one unit, or he can play on. If he does, the die is turned so that the "2" face is up, and he places the die on his side of the table. This symbolizes the rule that the doubler may not double again until his opponent has doubled. That is, the players must alternate doubling. At the nth double the nondoubler may either quit and pay the doubler 2^{n-1} units (fold) or he may play on with the 2^n face of the die up.

Now suppose A has the doubling die and his probability of winning = $p(t) = \alpha$. For which α should A double? Also, for which α should B resign if A doubles? There may be α at which A has equal expectations whether or not he doubles, and other α where B has equal expectations whether he accepts or folds. Assume A doubles when he may and B accepts when he may. The expected gain to player A is a monotone function of $p(t)$. Thus, there will be an α_0 such that A doubles if $\alpha > \alpha_0$ and doesn't if $\alpha < \alpha_0$. By continuity considerations, A can do either if $\alpha = \alpha_0$. So assume A doubles iff $\alpha \geq \alpha_0$. But then, since $p(t)$ is continuous, A's strategy is to double exactly at $\alpha = \alpha_0$, which we call A's doubling point. [Of course, if the game starts with $\alpha > \alpha_0$ or if $\alpha > \alpha_0$, after B doubles A should double immediately. Such contingencies can be easily taken care of but will be assumed not to occur here.] Similarly B has a folding point β_0.

<u>THEOREM 1</u>: $\alpha_0 = \beta_0$

<u>PROOF</u>: Since B is assumed perfect, A's expectation is $\leq$ the current value of the game. So if B would fold, A should double. Therefore A should double at β_0 since B will fold (or do equally poorly if he accepts). So $\alpha_0 \leq \beta_0$. Next, suppose that $\alpha_0 < \gamma < \beta_0$. Lets compare A's strategies of doubling at α_0 with doubling at γ. If A doubles at α_0 either the game will eventually reach state γ or it won't. If it does, A could have waited since B will accept anyway. If the game never reaches state γ, A loses, and loses more by his double at α_0. Therefore to double at $\alpha_0 < \beta_0$ is premature and thus $\alpha_0 \geq \beta_0$ so $\alpha_0 = \beta_0$.

Now we can solve for α_0. By symmetry B will double at state $1-\alpha_0$. When A doubles at α_0, B must have equal expectations whether he folds or accepts. Let K be the current stakes of the game. Then if B folds he gets $-K$. Assume he accepts. Applying our lemma, A will win before B gets a chance to double with probability $\frac{2\alpha-1}{\alpha}$. With probability $\frac{1-\alpha}{\alpha}$ B gets a chance to double and, since it doesn't affect the expectations, we can assume A resigns. So

(3) $$-K = 2K[\frac{1-\alpha}{\alpha} - \frac{2\alpha-1}{\alpha}]$$

(4) $$-\alpha = 2[2 - 3\alpha]$$

(5) $$\alpha = .8 = 80\%$$

Now suppose the doubling die is in the center. When player A doubles B can consider the situation as if A has had the doubling die. So B's folding point is 80%. Then the argument of Theorem 1 shows that A shouldn't double until $\alpha = 80\%$.

This completes the strategy. A player doubles iff he has an 80% chance of winning and accepts a double iff he has at least a 20% chance of winning. It is not hard to show that this optimal strategy is unique, as it has a positive expectation against any other strategy.

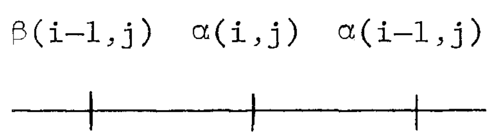
Suppose that the total number of raises by both players must be no greater than n. By similar arguments, we can show that the first doubling point $\alpha(n) = 4/5 + \frac{(-1)^n}{4^n \cdot 5}$.

3. NONALTERNATING RAISES

Let us suppose that the betting rules are changed. Player A is allowed to raise the stakes n times by a factor of a each time, and Player B is allowed to raise m times by a factor of b. The bets no longer have to alternate, and either player can raise whenever he pleases. For simplicity we will imagine the initial stakes to be one. Let $\alpha(i,j)$ represent the smallest raising point for Player A when he has i raises left and B has j. Just as in Section 2, $\alpha(i,j)$ is also the largest point at which player B should accept, and hence the point at which the expected loss to B equals the present value of the game. Let $\beta(i,j)$ be the largest doubling point for player B, when player A has i bets left and player B has j.

LEMMA: $\beta(i-i,j) < \alpha(i,j) \leq \alpha(i-i,j)$

PROOF: Since a player may always decide not to use a bet, his proportional expected gain is not less with i bets than with i–1. Since α is determined by the expected gain, this establishes the right hand inequality. Since player B's expected gain is negative at $\alpha(i,j)$ after the raise, he should not reraise immediately..



We will establish a recursive formula for $\alpha(i,j)$. Let the present value of the game be K. Since at the raise points the nonraiser is indifferent between folding and playing on, we can view the gams as ending at $\beta(i-1,j)$ or $\alpha(i-1,j)$. Thus

(6) $$-K = aK\left(\frac{\alpha(i-1,j)-\alpha(i,j)}{\alpha(i-1,j)-\beta(i-1,j)} - \frac{\alpha(i,j)-\beta(i-1,j)}{\alpha(i-1,j)-\beta(i-1,j)}\right)$$

B folds   B accepts and the game goes first to

$\beta(i-1,j)$ $\alpha(i-1,j)$

(7) $$-[\alpha(i-1,j) - \beta(i-1,j)] = a(\alpha(i-1,j) + \beta(i-1,j) - 2\alpha(i,j))$$

(8) $$\alpha(i,j) = \frac{a+1}{2a}\,\alpha(i-1,j) + \frac{a-1}{2a}\,\beta(i-1,j)$$

Similarly

(9) $$\beta(i,j) = \frac{b-1}{2b}\,\alpha(i,j-1) + \frac{b+1}{2b}\,\beta(i,j-1)$$

Since it doesn't effect the expectations, let us suppose all such raises are accepted. After A doubles at state $\alpha(i,j)$, A will get the next double with probability

(10) $$\frac{\alpha(i,j) - \beta(i-1,j)}{\alpha(i-1,j)-\beta(i-1,j)} = \frac{1+a}{2a}$$

Similarly, after B doubles, he will get the next double with probability $\frac{1+b}{2b}$. Suppose we start the game at state $\alpha(m,n)$. Then A's probability of winning (not his expected value) is $\alpha(m,n)$.

This game is equivalent to the following situation. A flips a biased coin, which comes up heads with probability $\frac{1+a}{2a}$. If he gets a head, the P-game goes from state $\alpha(i,j)$ to A's next doubling point $\alpha(i-1,j)$. If he gets a tail (an "upset"), the P-game goes to state $\beta(i-1,j)$ and B starts flipping his coin which comes up heads with probability $\frac{1+b}{2b}$. B flips until he gets an upset, then A flips, etc. A wins if he flips m times [reaching state $\alpha(0,j) = 1$] before B flips n times.

Now we reduce one more time. Suppose at the start of the game A flips his coin m times, and B his coin n times. Then A wins iff he has thrown less than or equal to as many tails as B. We state our result with an inductive proof.

<u>THEOREM 2</u>: Suppose, with the betting rules as described, A has m raises of factor a, and B has n raises of factor b. Let K_a, K_b be given by the probability distributions

$$\text{(11)}\quad \begin{aligned} \text{Prob } K_a &= b(m, \tfrac{a-1}{2a}, K_a) \\ \text{Prob } K_b &= b(n, \tfrac{b-1}{2b}, K_b) \end{aligned}$$

where $b(t,p,s)$ = the probability of exactly s successes in t trials where the probability of a single success is p. Then the proper doubling point $\alpha(m,n) = \text{Prob}[K_a \le K_b]$ and the doubling point $\beta(m,n)$ for B is $\text{Prob}[K_a < K_b]$.

<u>PROOF</u>: We need show this formula satisfies the inductive formula for $\alpha(i,j)$ and $\beta(i,j)$ and the initial conditions $\alpha(0,j) = 1$,

$\beta(i,0) = 1$. The initial conditions are straightforward. Now we need show

$$(8) \qquad \alpha(i,j) = \frac{a+1}{2a}\,\alpha(i-1,j) + \frac{a-1}{2a}\,\beta(i-1,j).$$

(The proof for (9) is similar.) Let K_a, K_b be the probability distributions for $m = i$, $n = j$, K_a', K_b' the distributions for $m = i-1$, $n = j$. So we want

$$\text{Prob}[K_a \leq K_b] = \frac{a+1}{2a}\,\text{Prob}[K_a' \leq K_b'] + \frac{a-1}{2a}\,\text{Prob}[K_a' < K_b']$$

The variable K_a consists of one toss plus the variable K_a'. $K_a \leq K_b$ iff the first toss is tails and $K_a' \leq K_b'$ or the first toss is heads and $K_a' < K_b'$. This proves the above equation and therefore the induction.

Example: Let us suppose the raising factor is a double, that $a = b = 2$. Tables I and II give the proper doubling points and expected value for A when he has i and B has j doubles left. If, for example, A starts with 10 doubles and B with 3, A should double at .4938 and therefore should double at the start of an even game.

Example: It may be helpful to look at a simple game. Suppose A has two raises of size 2, B has one raise of size 3. The tree shows all the relevant possibilities after the game goes to $\alpha(2,1)$

In the figure, the value to player A is written above each node. The "state" of the game is written below. The numbers on the edges are the probabilities of each path, needed to make the values consistent (cf. formulas 8, 9). For example, $\beta(0, 1) = \frac{1}{3}\,\alpha(0,0) + 2/3\ \beta(0,0)$ since $-4 = 1/3\ (12) + 2/3\ (-12)$. $\alpha(0,k)$ is one, since if A has no more raises, the effective end of the game for B is the real end.

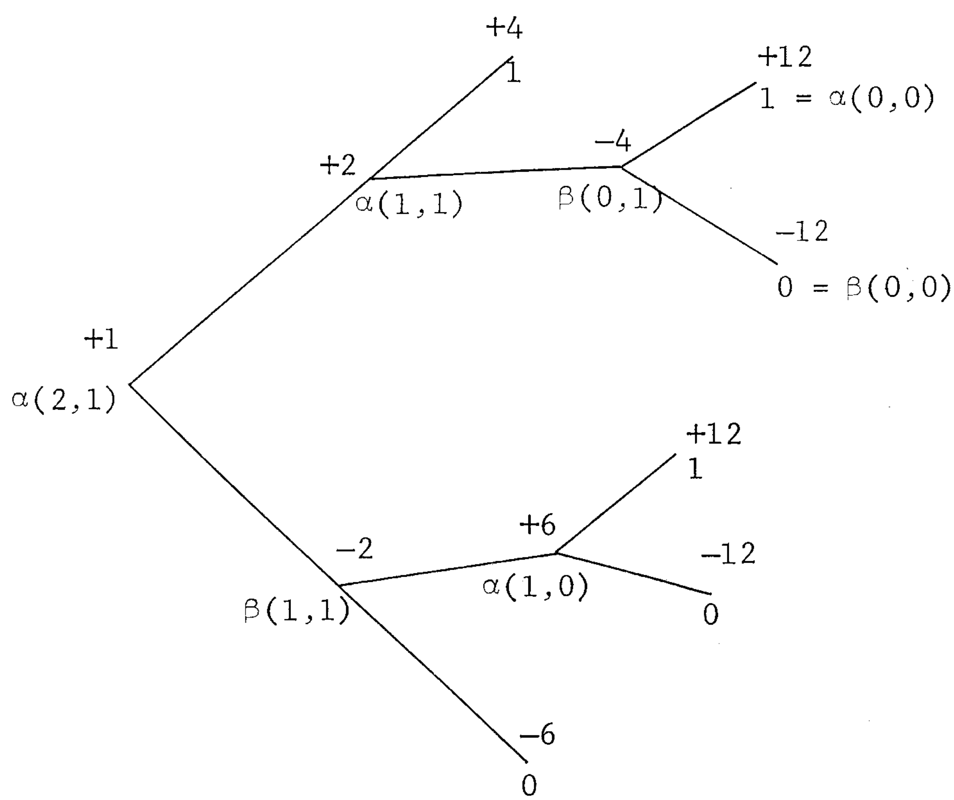


Figure 1

The values are only used in computing the edge probabilities, so we may use the figure below to compute $\alpha(2,1)$. Working backward, we get $\beta(0,1) = 1/3$, $\alpha(1,1) = 3/4 + 1/4 \cdot 1/3 = 5/6$ $\alpha(1,0) = 3/4$ $\beta(1,1) = 1/3 \cdot 3/4 + 2/3 \cdot 0 = 1/4$. Finally $\alpha(2,1) = 3/4 \cdot 5/6 + 1/4 \cdot 1/4 = \frac{11}{16}$. We can see the explanation for Theorem 2 more clearly if we ignore the intermediate nodes. Thus $\alpha(2,1) = 3/4 \cdot 1/4 \cdot 1/3 + 1/4 \cdot 1/3 \cdot 3/4 =$ prob $K_1 \leq K_2$ where $P(K_1) = b(2, 1/4, K_1)$ and prob $K_2 = b(1, 1/3, K_2)$.

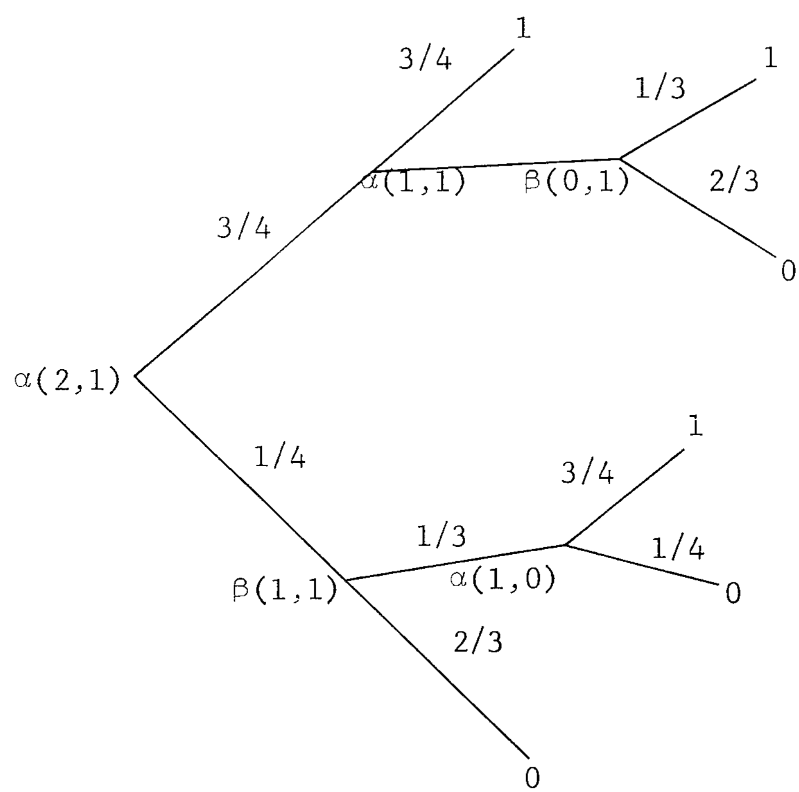


Figure 2

4. CONTINUOUS RAISES

Suppose that the restraints on raising are relaxed further. Each player has a maximum raise factor, but at any time he may raise the value of the game by any factor greater than one, provided the product of his raises does not exceed his maximum. Let player A's maximum factor be a, player B's b. We will compute $\alpha(a,b)$ by considering a game in which A has n raises of $a^{1/n}$, b has m raises of $b^{1/m}$ and finding the limit as n and m approach infinity.

From section 3, we know that $\alpha(n,m) = \text{Prob }(K_a \leq K_b)$ where

(12) $$\text{Prob }(K_a) = b(n, \frac{a^{1/n} - 1}{2a^{1/n}}, K_a)$$

(13) $$\text{Prob }(K_b) = b(m, \frac{b^{1/m} - 1}{2b^{1/m}}, K_b)$$

Now

(14) $$\lim_{n\to\infty} \frac{n(a^{1/n} - 1)}{2a^{1/n}} = \frac{1}{2} \lim_{n\to\infty} \frac{a^{1/n} - 1}{1/n} = \text{(by L'Hôpitals rule)}$$

$$\frac{1}{2} \lim_{n\to\infty} \frac{a^{1/n}(\frac{-1}{n^2} \log a)}{-1/n^2} = \frac{1}{2} \log a.$$

Similarly

(15) $$\lim_{m\to\infty} \frac{m(b^{1/m} - 1)}{2b^{1/m}} = \frac{1}{2} \log b$$

Therefore, as $n \to \infty$, Prob (K_a) approaches $P[\frac{1}{2} \log a, K_a]$, the probability of K_a successes in a Poisson distribution of mean $\frac{1}{2} \log a$. Similarly with $P(K_a)$. Setting

(16)
$$L = \frac{1}{2} \log a$$
$$M = \frac{1}{2} \log b$$

we have

(17)
$$P(K_a) = \frac{e^{-L} L^{K_a}}{K_a!} = \frac{L^{K_a}}{\sqrt{a}\, K_a!}$$

and therefore, if we set $\alpha(a,b)$ = the raising point for A when A has a total factor of a [and B has b], we have

(18)
$$\alpha(a,b) = \frac{1}{\sqrt{ab}} \sum_{\substack{i \leq j \\ i,j=0}}^{\infty} \frac{L^i M^j}{i!\,j!}$$

Example: If $b = 1$, $M = 0$, $\alpha(a,b) = \frac{1}{\sqrt{a}} \sum_{i \leq j} \frac{L^i 0^j}{i!\,j!} = \frac{1}{\sqrt{a}}(1)$ [since $i = j = 0$ gives the only positive term] $= \frac{1}{\sqrt{a}}$. Thus if $a > 4$ and the game is fair (i.e. $p(0) = \frac{1}{2}$) A should raise immediately and B should resign.

With a discrete number of raises we reduced the game to a coin flipping contest. We have a similar analogy here with chess clocks [this is a pair of clocks so set up that exactly one of the clocks is running an any time].

Suppose we start the game at A's first raising point. His clock runs until an "upset" [Poisson with mean $\frac{1}{2}$ log a] occurs. Then B's clock runs until an upset [Poisson with mean $\frac{1}{2}$ log b] occurs. Both clocks are initially set at one time unit. A wins iff his clock runs out before B's does. If b = 1, A wins iff A's clock runs out before B's gets a chance to run. A's clock running time t represents a raise by A of factor a^t.

5. ASYMPTOTIC BEHAVIOR

The example at the end of section 4 showed that if player A had a raise factor of 4 while player B had a raise factor of 1, then A's expected gain was 1 at p = 1/2. If B was very large, how much bigger would A have to be to dominate the game in this way? In general, what can we say about the asymptotic (in terms of raise size) behavior of the game. A more rigorous confirmation of the results below is found in the appendix.

The expected gain is found by the formula

$$(19) \quad \text{E Gain (A)} = \frac{(\frac{1}{2} - \beta(a,b)) - (\alpha(a,b) - \frac{1}{2})}{\alpha(a,b) - \beta(a,b)} = \frac{1-\beta(a,b) - \alpha(a,b)}{\alpha(a,b) - \beta(a,b)}$$

provided $\beta(a,b) < \frac{1}{2} < \alpha(a,b)$. If this condition is not fulfilled one of the players dominates the game and his expected gain is 1. By symmetry $\beta(a,b) = 1 - \alpha(b,a)$ so (19) becomes

$$(20) \quad \text{E Gain (A)} = \frac{\alpha(b,a) - \alpha(a,b)}{\alpha(a,b) + \alpha(b,a) - 1}$$

Let

$$(21) \quad L = \frac{1}{2} \log a$$

$$M = \frac{1}{2} \log b$$

so

(22) $\alpha(a,b) = \text{Prob } (X \leq Y)$ where $X: P(L,X)$

$= \text{Prob } (X-Y \leq 0)$ $Y: P(M,X)$

If $L,M \gg 0$ then X(resp. Y) can be approximated by a normal distribution of mean L(M) and variance L(M). Therefore $Z = X-Y$ is approximately a normal distribution with mean $L-M$ and variance $L+M$. Now $\alpha(a,b) \simeq \text{Prob } (Z \leq 0)$. But we've estimated our discrete Z by a continuous one so the proper approximation is

(23) $$\alpha(a,b) \sim \text{Prob } (Z < \tfrac{1}{2}) = \Phi\left(\frac{\frac{1}{2} - (L-M)}{\sqrt{L+M}}\right)$$

where $\Phi(t) = \int_{-\infty}^{t} e^{-x^2/2}dx$ is the normal distribution. This suggests setting

(24) $$L = M + t$$

So then

(25) $$\text{E Gain (A)} \sim \frac{\Phi\left(\frac{\frac{1}{2} + t}{\sqrt{2m+t}}\right) - \Phi\left(\frac{\frac{1}{2} - t}{\sqrt{2m+t}}\right)}{\Phi\left(\frac{\frac{1}{2} + t}{\sqrt{2m+t}}\right) + \left(\frac{\frac{1}{2} - t}{\sqrt{2m+t}}\right) - 1}$$

Let t be fixed and $M \to \infty$. Both numerator and denominator approach 0. Since $\Phi'(u) = e^{-u^2} \cdot u'$, we get by L'Hopital's rule,

$$\lim_{m\to\infty} \text{E Gain (A)} = \lim_{m\to\infty} \frac{c-d}{c+d}$$

where $c = \dfrac{e^{-\frac{1}{2}(\frac{1}{2}+t)^2/(2m+t)}\,(\frac{1}{2}+t)(\frac{-1}{2})2}{(2m+t)^{3/2}}$

and $d = c$ with the factor of $(\frac{1}{2} + t)$ changed to $(\frac{1}{2} - t)$

As $M \to \infty$ the exponential factors approach unity, the denominators approach each other, so

$$(26) \quad \lim_{m\to\infty} \text{E Gain (A)} = \lim_{m\to\infty} \frac{-(\frac{1}{2}+t) + (\frac{1}{2}-t)}{-(\frac{1}{2}+t) - (\frac{1}{2}-t)} = \frac{-2t}{-1} = 2t = \log\frac{a}{b}\ .$$

Results: If $a = b$, E Gain (A) $= 0$

If $a = be$, $b \gg 1$, then E Gain (A) ~ 1

If $a = b, \alpha(a,b) \sim \Phi(\frac{1}{2\sqrt{2L}} \sim \frac{1}{2} + \frac{1}{2\sqrt{\log a}})$

so $\alpha(a,a) \to \frac{1}{2}$ quite slowly.

In Tables III and IV we give the raising points and the expected value of the game starting with an even game, where the raise potentials a,b = 1(1)10. In Table V and Fig. 3 we give, and graph, as a function of B's raise potential b, what A's raise potential a must be for A to have expected value + 1. Note $a \sim be$, approaching it from above.

Also note the shift in values from the discrete to the continuous case. For example, in Table II, if A has

3 doubles and B has 2, A has an expected value of + .430. But in Table IV, if A has a raise factor of 8 and B of 4, A has an expected value of + .581. It seems that, in general, making the raises "continuous" helps the player with positive expectation. An extreme example would be when A has a raise factor of 10^{10} and B has no raise factor. If A may raise only once he must wait until he has a 50% chance of winning. But if he may raise continuously he may force B out if he has a $1/\sqrt{10^{10}} = 10^{-5}$ probability of winning.

APPENDIX

THEOREM: Let E(a,b) = the expected gain for the first player if he starts with a factor of a and the second starts with a factor of b. We assume the two doubling points lie on different sides of 1/2. Then

$$(27)\qquad E(a,b) = \log \frac{a}{b} + 0(1)$$

where $0(1) \to 0$ as $a,b \to \infty$.

PROOF: Setting $M = \frac{1}{2}\log b$, $M + t = \frac{1}{2}\log a$ the exact formula is

$$(28)\qquad E(a,b) = \frac{\sum\limits_{i>j} \frac{(m+t)^i M^j}{i!j!} - \sum\limits_{i<j} \frac{(M+t)^i M^j}{i!j!}}{\sum \frac{(M+t)^i M^i}{i!i!}}$$

We assume t remains fixed as $M \to \infty$.

Let us set $f(x) = \sum \frac{x^{2i}}{i!^2}$. By advanced calculus,* $f(x) = I_0(2x)$ [where I_0 is the modified Bessel function of the first kind of order 0] so $f(x) \sim \frac{e^{2x}}{2\sqrt{\pi x}}$. Therefore the denominator $= f(\sqrt{M(M+t)}) = f(M + \frac{t}{2} + 0(1)) \sim \frac{e^{2m+t}}{2\sqrt{\pi M}}$.

Expanding the numerator in powers of t gives

$$\text{Numerator} = \sum \frac{C_k t^k}{k!}$$

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*e.g.: Hildebrand, "Advanced Calculus for Engineers," p. 155.

where $C_k = \sum_{\substack{j<i \\ k\leq i}} \frac{M^{j+(i-k)}}{j!(i-k)!} - \sum_{\substack{i<j \\ k\leq i}} \frac{M^j M^{i-j}}{j!(i-j)!}$

Setting $i = i-k$

$$C_k = \sum_{j<i+k} \frac{M^i M^j}{i!j!} - \sum_{i+k<j} \frac{M^i M^j}{i!j!}$$

$$= \sum_{j=i} \frac{M^i M^j}{i!j!} + 2 \sum_{i<j<i+k} \frac{M^{i+j}}{i!j!} + \sum_{j=i+k} \frac{M^i M^j}{i!j!}$$

So C_k is bounded by e^{2M}. Therefore the k^{th} term, after division by the denominator is less than approximately

$$\frac{2\sqrt{\pi M}\ t^k}{e^t k!}$$

Now t is fixed, and $M \gg t$, so if $k = \log M$, $k! \sim M^{(\log\log M)-1}$ so for M very large this term, and the summation for $k \geq \log M$ approaches zero. Therefore we need consider only the summation from $k = 1$ to $\log M$. For $k = 1$, $\sum_{j=i} \frac{M^i M^j}{i!j!} = f(M)$.

Now if $k < \log M$ by comparing terms we can show $\sum_{j=i+k} \frac{M^i M^j}{i!j!} \sim f(M)$. Therefore

$$C_k \sim 2kf(M)$$

$$\text{Numerator} \sim \sum \frac{2kt^k}{k!} f(M) = 2tf(M)e^t$$

So

$$E(a,b) \sim \frac{2tf(M)e^{t}}{f(\sqrt{M(M+t)})} \sim \frac{2te^{t}e^{2M}/2\sqrt{\pi M}}{e^{2M+t}/2\sqrt{\pi M}} \sim 2t = \log \frac{a}{b} .$$

DOUBLING POINTS AS A FUNCTION OF NUMBER OF DOUBLES

	0	1	2	3	4	5	6	7	8	9
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	.7500	.8125	.8594	.8945	.9209	.9407	.9555	.9666	.9750	.9812
2	.5625	.6563	.7305	.7891	.8352	.8715	.8999	.9221	.9395	.9531
3	.4219	.5273	.6152	.6880	.7479	.7969	.8369	.8693	.8955	.9167
4	.3164	.4219	.5142	.5940	.6624	.7205	.7696	.8107	.8451	.8736
5	.2373	.3362	.4268	.5085	.5812	.6451	.7008	.7488	.7900	.8252
6	.1780	.2670	.3522	.4322	.5058	.5726	.6324	.6856	.7323	.7731
7	.1335	.2113	.2892	.3649	.4369	.5042	.5662	.6226	.6734	.7187
8	.1001	.1669	.2364	.3064	.3750	.4410	.5033	.5613	.6147	.6634
9	.0751	.1314	.1924	.2559	.3200	.3832	.4444	.5026	.5574	.6082

TABLE I

EXPECTED VALUES AS A FUNCTION OF NUMBER OF DOUBLES

	0	1	2	3	4	5	6	7	8	9
0	.000	-.333	-.778	-1.000	-1.000	-1.000	-1.000	-1.000	-1.000	-1.000
1	.333	.000	-.394	-.870	-1.000	-1.000	-1.000	-1.000	-1.000	-1.000
2	.778	.394	.000	-.430	-.919	-1.000	-1.000	-1.000	-1.000	-1.000
3	1.000	.870	.430	.000	-.450	-.944	-1.000	-1.000	-1.000	-1.000
4	1.000	1.000	.919	.450	.000	-.462	-.958	-1.000	-1.000	-1.000
5	1.000	1.000	1.000	.944	.462	.000	-.469	-.967	-1.000	-1.000
6	1.000	1.000	1.000	1.000	.958	.469	.000	-.474	-.972	-1.000
7	1.000	1.000	1.000	1.000	1.000	.967	.474	.000	-.478	-.976
8	1.000	1.000	1.000	1.000	1.000	1.000	.972	.478	.000	-.480
9	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.976	.480	.000

TABLE II

RAISING POINTS AS A FUNCTION OF RAISE POTENTIAL

	1	2	3	4	5	6	7	8	9	10
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	.7071	.7809	.8153	.8363	.8510	.8620	.8707	.8778	.8837	.8888
3	.5774	.6745	.7209	.7498	.7703	.7857	.7980	.8081	.8166	.8239
4	.5000	.6074	.6597	.6927	.7161	.7340	.7483	.7601	.7700	.7786
5	.4472	.5598	.6153	.6507	.6760	.6954	.7110	.7238	.7347	.7441
6	.4082	.5235	.5810	.6179	.6445	.6649	.6813	.6949	.7065	.7164
7	.3780	.4945	.5534	.5913	.6187	.6398	.6568	.6710	.6830	.6934
8	.3536	.4707	.5303	.5690	.5970	.6186	.6361	.6507	.6631	.6738
9	.3333	.4506	.5107	.5499	.5783	.6004	.6182	.6331	.6458	.6568
10	.3162	.4333	.4938	.5332	.5620	.5844	.6025	.6176	.6305	.6418

Table III

EXPECTED VALUES AS A FUNCTION OF RAISE POTENTIAL

	1	2	3	4	5	6	7	8	9	10
1	.000	-.414	-.732	-1.000	-1.000	-1.000	-1.000	-1.000	-1.000	-1.000
2	.414	.000	-.287	-.516	-.709	-.878	-1.000	-1.000	-1.000	-1.000
3	.732	.287	.000	-.220	-.402	-.558	-.696	-.821	-.934	-1.000
4	1.000	.516	.220	.000	-.178	-.330	-.462	-.581	-.688	-.787
5	1.000	.709	.402	.178	.000	-.150	-.280	-.395	-.500	-.595
6	1.000	.878	.558	.330	.150	.000	-.129	-.243	-.346	-.439
7	1.000	1.000	.696	.462	.280	.129	.000	-.114	-.215	-.307
8	1.000	1.000	.821	.581	.395	.243	.114	.000	-.101	-.193
9	1.000	1.000	.934	.688	.500	.346	.215	.101	.000	-.091
10	1.000	1.000	1.000	.787	.595	.439	.307	.193	.091	.000

TABLE IV

TABLE V: RAISE POTENTIALS SUCH THAT B SHOULD RESIGN.

b	a	b	a	b	a
1.0	4.0	4.7	14.4316	15.5	45.0150
1.1	4.2775	4.8	14.7151	16.0	46.4280
1.2	4.5556	4.9	14.9987	16.5	47.8407
1.3	4.8342	5.0	15.2822	17.0	49.2533
1.4	5.1132	5.1	15.5658	17.5	50.6655
1.5	5.3927	5.2	15.8494	18.0	52.0776
1.6	5.6725	5.3	16.1329	18.5	53.4895
1.7	5.9527	5.4	16.4165	19.0	54.9011
1.8	6.2332	5.5	16.7001	19.5	56.3126
1.9	6.5141	5.6	16.9837	20.0	57.7238
2.0	6.7952	5.7	17.2673	25	71.827
2.1	7.0765	5.8	17.5508	30	85.914
2.2	7.3581	5.9	17.8344	35	99.990
2.8	9.0510	6.0	18.1180	40	114.055
2.9	9.3336	6.5	19.5359	45	128.112
3.0	9.6164	7.0	20.9536	50	142.163
3.1	9.8992	7.5	22.3712	55	156.208
3.2	10.1821	8.0	23.7885	60	170.248
3.3	10.4650	8.5	25.2056	65	184.283
3.4	10.7481	9.0	26.6224	70	198.315
3.5	11.0312	9.5	28.0389	75	212.344
3.6	11.3144	10.0	29.4551	80	226.371
3.7	11.5976	10.5	30.8710	85	240.395
3.8	11.8808	11.0	32.2867	90	254.417
3.9	12.1641	11.5	33.7020	95	268.437
4.0	12.4475	12.0	35.1171	100	282.456
4.1	12.7308	12.5	36.5319	200	562.752
4.2	13.0142	13.0	37.9464	500	1405.576
4.3	13.2977	13.5	39.3607	1000	2819.971
4.4	13.5811	14.0	40.7746		
4.5	13.8646	14.5	42.1884		
4.6	14.1481	15.0	43.6018		

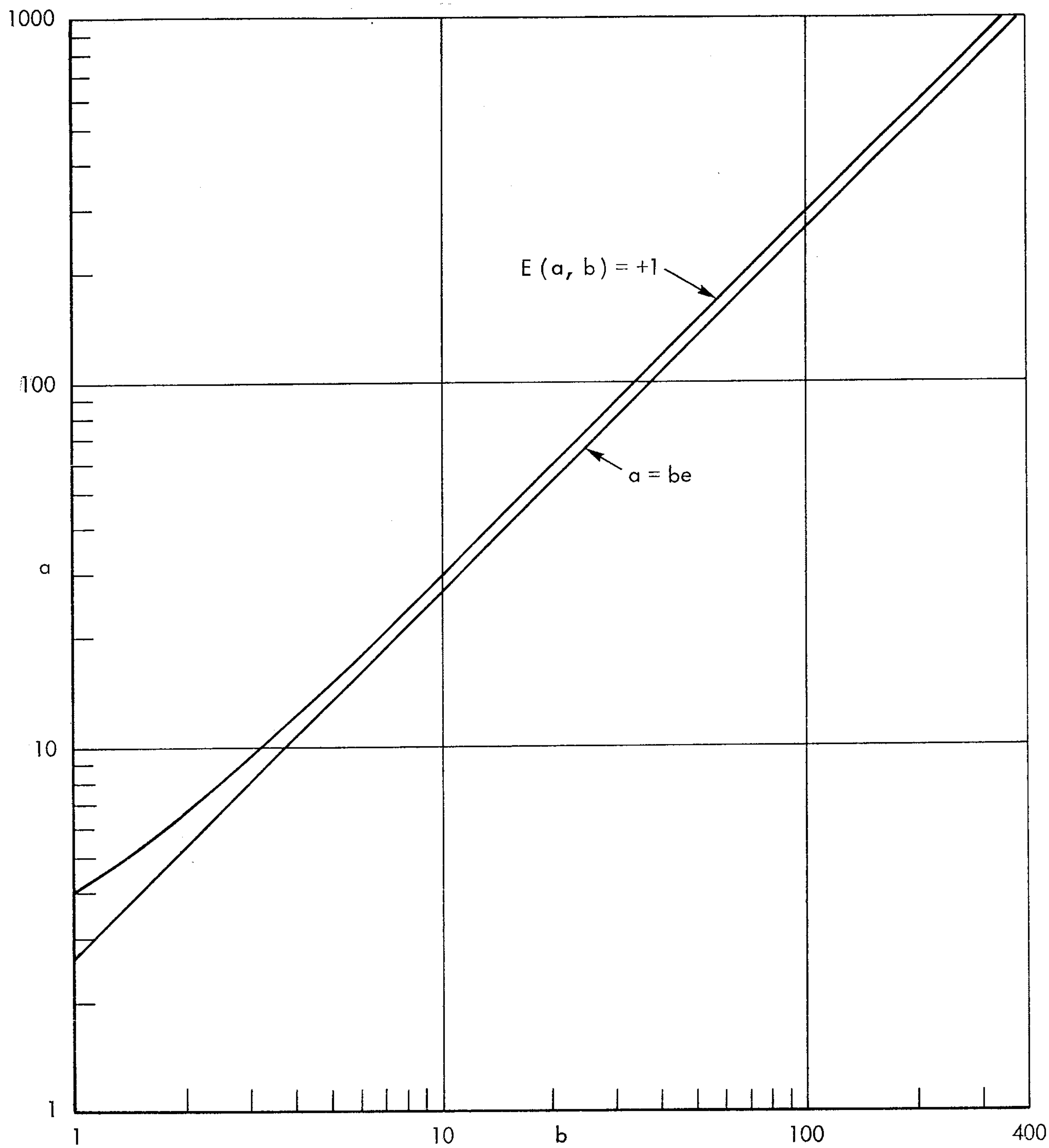

Figure 3

P-4078 PROPER RAISING POINTS IN A GENERALIZATION OF BACKGAMMON E. Keeler and J. Spencer